MA 410 Tuts

Tutorial-1 :

1. $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation

$|T(x)| = |x| \;\forall\, x \in \mathbb{R}^n$
this is called norm preserving

(a) To prove : T is 1-1

proof : Given T is a linear trasformation and

$$|T(x)| = |x| = \sqrt{\Sigma (x^i)^2}$$

for
$$T(x) = T(y)$$
$$\Rightarrow T(x-y) = T(0)$$
$$T(x-y) = 0$$
$$\Rightarrow |T(x-y)| = 0$$
$$\Rightarrow |x-y| = 0$$
$$\Rightarrow x-y = 0$$
$$\Rightarrow x = y$$

$\therefore$ T is 1-1

(b) To prove : T^{-1} exist

proof : If T is onto then T^{-1} exist,
as by Rank nullity

$$\text{Dim}(\mathbb{R}^n) = \text{Rank}(T) + \text{Null}(T)$$
and as
$$n = \text{Rank}(T) + 0$$
as T is 1-1
$$\Rightarrow n = \text{Rank}(T)$$
Ras $\text{Ran}(T) \subseteq \mathbb{R}^n$
but
$\dim(\mathbb{R}^n) = n$
$\Rightarrow \text{Ran}(T) = \mathbb{R}^n$
$\Rightarrow$ T is onto
as Range = co-domain

$\therefore T^{-1}$ exist

(c) To prove : T^{-1} is norm preserving

Proof : To show that $|T^{-1}(y)| = |y|$

as $|T(x)| = |x|$
and $x = T^{-1}(y)$
$$\Rightarrow |T(T^{-1}(y))| = |T^{-1}(y)|$$
$$\Rightarrow |y| = |T^{-1}(y)|$$

2. To prove : $x = (x^1, x^2, \ldots x^n) \in \mathbb{R}^n$

$$|x| \leq \sum_{i=1}^{n} |x^i| \leq \sqrt{n}\,|x|$$

Proof : Here this is same as
$$c_1 |x|_1 \leq |x|_2 \leq c_2 |x|_1$$

where $|\sum x^i y^i| \leqslant |x||y|$

$$|x| = \sqrt{\sum (x^i)^2}$$

$$\sum |x^i| = |x^1| + |x^2| + \cdots |x^n|$$

as $\quad 2|x^i||x^j| \geqslant 0$

n-1 times

$$\sum |x^i|^2 + 2\sum_{i>j} |x^i||x^j| \leqslant \sum |x^i|^2 + \sum_{i>j} |x^i|^2 + |x^j|^2 \quad = n \sum |x^i|^2$$

n-1 times

by AP $\geqslant$ GP
$|x^i|^2 + |x^j|^2 \geqslant 2|x^i||x^j|$

$$\Rightarrow \left(\sum |x^i|\right)^2 \leqslant n\left(\sum |x^i|^2\right)$$

$$\Rightarrow \sum |x^i| \leqslant \sqrt{n}\, |x|$$

3. $x = (x^1, x^2)$
$y = (y^1, y^2)$

$$\langle\langle x, y \rangle\rangle = [x^1 \quad x^2]_{1\times 2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$$

To prove: $\langle\langle x, y \rangle\rangle$ is an inner product

proof: ① symmetry:

$$\langle\langle x, y \rangle\rangle = \begin{bmatrix} 2x^1 - x^2 & -x^1 + x^2 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$$

$$= 2x^1y^1 - x^2y^1 - x^1y^2 + x^2y^2$$

$$= 2x^1y^1 - x^2y^1 - x^1y^2 + x^2y^2$$

$$\langle\langle y, x \rangle\rangle = 2y^1x^1 - y^2x^1 - y^1x^2 + y^2x^2$$

$$= \langle\langle x, y \rangle\rangle$$

② Bilinear:

$$\langle\langle ax, y \rangle\rangle = 2ax^1y^1 - ax^2y^1 - ax^1y^2 + ax^2y^2$$
$$= a\langle\langle x, y \rangle\rangle$$

same for $\langle\langle x, ay \rangle\rangle$

$$\langle\langle x_1 + x_2, y \rangle\rangle = 2(x_1^1 + x_2^1)y^1 - (x_1^2 + x_2^2)y^1$$
$$- (x_1^1 + x_2^1)y^2 + (x_1^2 + x_2^2)y^2$$

$$= \langle\langle x_1, y \rangle\rangle + \langle\langle x_2, y \rangle\rangle$$

same for $\langle\langle x, y_1 + y_2 \rangle\rangle$

③ position: $\langle\langle x, x \rangle\rangle = 2(x^1)^2 - x^1x^2 - x^1x^2 + (x^2)^2$
$$= 2(x^1)^2 - 2(x^1x^2) + (x^2)^2$$

$$\text{as } \left[(\sqrt{2}x^1)^2 (x^2)^2\right]^{1/2} \leq \frac{(\sqrt{2}x^1)^2 + (x^2)^2}{2}$$

$$\Rightarrow \quad 2(x^1)(x^2) \leq \frac{2(x^1)^2 + (x^2)^2}{\sqrt{2}}$$

$$\Rightarrow \quad -2x^1x^2 \geq -\frac{2(x^1)^2 - (x^2)^2}{\sqrt{2}}$$

$$\Rightarrow 2(x^1)^2 + (x^2)^2 - 2x^1x^2 \geq \frac{\sqrt{2}}{\sqrt{2}}\left(2(x^1)^2 + (x^2)^2\right) - \frac{(2x^{1^2} + x^{2^2})}{\sqrt{2}}$$

$$\geq 0$$

and if $\langle\langle x, x\rangle\rangle = 0 = 2(x^1)^2 + (x^2)^2 - 2x^1x^2$

$$2(x^1)^2 + (x^2)^2 = 2x^1x^2$$

if $x^2 \neq 0$

then

$$2\left(\frac{x^1}{x^2}\right)^2 + 1 = 2\left(\frac{x^1}{x^2}\right)$$

$$2\left(\frac{x^1}{x^2}\right)^2 - 2\left(\frac{x^1}{x^2}\right) + 1 = 0$$

$$\Delta = 4 - 4(2)(1) < 0$$

no solution

$\therefore x_2 = 0$

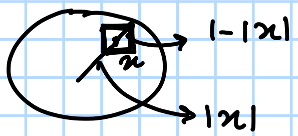
now for $x_2 = 0 \Rightarrow \quad x_1 = 0$

$\therefore \langle\langle x, x\rangle\rangle = 0$

$\Rightarrow x = 0$

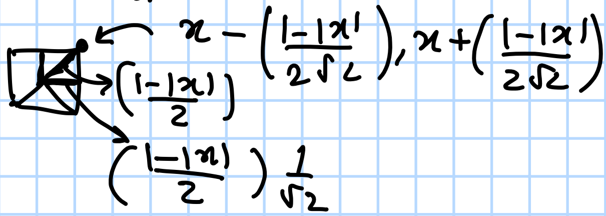
and $x = 0 \Rightarrow \langle\langle x, x\rangle\rangle = 0$ is trivial

4. $A = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$



interior: $A_i = \{x \in \mathbb{R}^2 \mid |x| < 1\}$

$$B = \left(x - \left(\frac{1-|x|}{2\sqrt{2}}\right), x + \left(\frac{1-|x|}{2\sqrt{2}}\right)\right)^2$$



then

$\forall x \in A_i$, this $\exists B$ s.t

$x \in B$ and

$B \subseteq A_i$

Exterior $A_\varepsilon = \{x \in \mathbb{R}^2 \mid |x| > 1\}$ as

for $\mathbb{R}^n \setminus A = |x| > 1$

and as it is open

$\exists$ Rectangles

Boudry: $AB = \{x \in \mathbb{R}^n \mid |x|=1\}$ as $|x|=1$ a unit square around the point has points s.t
$B \cap A \neq \phi$
and $B \cap (\mathbb{R}^n \setminus A) \neq \phi$

$B = \{x \in \mathbb{R}^n \mid |x| = 1\}$

Interior: No interior as any open netangle, the 4 cones will be outside B

Exterior: for this as $\mathbb{R}^n \setminus B$ is open, it is the exterior

Boudry: $B = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is the Boundary as all netengles will be s.t
netange $\cap (\mathbb{R}^n \setminus B) \neq \phi$

$C = \{x \in \mathbb{R}^n \mid x_i \text{ is rational}\}$

Interior and exterior will be empty as reet. will have both rational and irrational points.

The Boudry is $\mathbb{R}^n$ (By definafion) ← This defination should be carefully seen

5. $o(f, x_i) = \lim_{\delta \to 0} M(x_i, f, \delta) - m(x_i, f, \delta)$
by WOP orders $x_1, x_2 \ldots x_n$
as f is inc s.t $x_1 < x_2 < \cdots < x_4$

$m(x_{i+1}, f, \delta) > M(x_i, f, \delta)$
for $\delta \to 0$

$-m(x_{i+1}, f, \delta) < -M(x_i, f, \delta)$
$\Rightarrow M(x_{i+1}, f, \delta) - m(x_{i+1}, f, \delta)$
$< M(x_{i+1}, f, \delta) - M(x_i, f, \delta)$

$\Rightarrow o(x_{i+1}, f) < \lim_{\delta \to 0} M(x_{i+1}, f, \delta) - M(x_i, f, \delta)$

$$\sum_{i=0}^{n-1} o(x_{i+1}, f) = \sum_{i=1}^{n} o(x_i, f) < \lim_{\delta \to 0} M(x_n, f, \delta) - M(x_1, f, \delta)$$

as $f(b) \geq f(x)\ \forall x \in \text{Domain}$

$M(x_n, f, \delta) \leq f(b)$
and
sim $-M(x_n, f, \delta) \leq -f(a)$

$$\Rightarrow \sum_{i=1}^{n} o(x_i, f) < f(b) - f(a)$$

6. for $x \neq 0$ as

$$f(x,y) = x^3 \sin\left(\frac{y}{x}\right)$$

here x^3 is cont, $\sin(z)$ is cont, $\frac{y}{x}$ is cont

so $f(x,y)$ is cont as $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

for $x=0$: let's use epsilon delta defination

$|x-0|<\delta$ or $|x|<\delta$

then

$$|f(x,y) - f(0,y)| = \left|x^3 \sin\frac{y}{x}\right| < |x^3| < |\delta|^3$$

so for $\varepsilon = \delta^3 > 0$

s.t

$|x| < \delta \Rightarrow$

$|f(x,y) - f(0,y)|$

$< \varepsilon$

$\therefore f(x,y)$ is cont

Tutorial-2 :

1. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^3$

$$Df(a) = [3a^2]_{1\times 1} = T$$

then $T(h) = 3a^2 h \quad \forall h \in \mathbb{R}$

let's prove $Df(a) = [3a^2]$ first

① T is a linear map as

$$T(\alpha a + \beta b) = [3a^2](\alpha a + \beta b)$$
$$= \alpha[3a^2](a) + \beta[3a^2](b)$$
$$= \alpha T(a) + \beta T(b)$$
$$\forall \alpha, \beta, a, b \in \mathbb{R}$$

② $$\lim_{h\to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|}$$

$$= \lim_{h\to 0} \left|\frac{(a+h)^3 - (a)^3 - 3(a)^2 h}{|h|}\right|$$

$$= \lim_{h\to 0} \left|\frac{\cancel{a^3} + h^3 + 3ah^2 + \cancel{3a^2h} - \cancel{a^3} - \cancel{3a^2h}}{|h|}\right|$$

$$= \lim_{h\to 0} \frac{|h||h||h+3a|}{|h|} \to 0 \text{ as } h \to 0$$

$\therefore Df(a) = 3a^2$

now, (i) $a = 0$
$Df(0) = [0]_{1\times 1}$ $\quad Df(0)(x) = 0$

(ii) $a = 1$
$Df(1) = [3]_{1\times 1}$ $\quad Df(1)(x) = 3x$

2. $f: \mathbb{R}^n \to \mathbb{R}$ s.t

$$|f(x)| \leq |x|^2$$

To prove: f is diff at 0.

Proof: let $Df(0) = T = (0, 0, \dots 0)_{1\times n}$ is a trivial linear transform.

$|f(0)| \leq 0 \Rightarrow f(0) = 0$

then here $$\lim_{h\to 0} \frac{|f(0+h) - f(0) - T(h)|}{|h|}$$

$$= \lim_{h\to 0} \frac{|f(h) - T(h)|}{|h|}$$

$$\leq \lim_{h\to 0} \frac{|f(h)|}{|h|} + \frac{|T(h)|}{|h|}$$

$$\leq \underbrace{\lim_{h\to 0} \frac{|h|^2}{|h|}}_{\to 0} + \lim_{h\to 0} \frac{|T(h)|}{|h|}$$

$$T(h) = (0,0,\ldots 0)_{1\times n} \begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^n \end{pmatrix}_{n\times 1} = 0$$

then $T(h) = 0 \Rightarrow \lim_{h\to 0} \frac{|T(h)|}{|h|} = 0$

$$\therefore \lim_{h\to 0} \frac{|f(h) - f(0) - T(h)|}{|h|} = 0$$

$\therefore \exists$ linear transformation T s.t the above happens, so f is diff at 0.

3. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad f(x,y) = \sqrt{|xy|}$

<u>To prove</u>: f is <u>not</u> differentiable at $(0,0)$.

<u>proof</u>: let's assume f is differentiable at $(0,0)$, then by defination $\exists T$ (a linear map) $: \mathbb{R}^2 \longrightarrow \mathbb{R}$ s.t

$$\lim_{h\to 0} \frac{|f(0+h) - f(0) - T(h)|}{|h|} = 0$$

$$\Rightarrow \lim_{h\to 0} \left| \frac{\sqrt{|h^1 h^2|} - T(h)}{|h|} \right| = 0$$

$T = [T_1 \quad T_2]_{1\times 2}$ by defination

$$T(h) = [T_1 \quad T_2] \begin{pmatrix} h^1 \\ h^2 \end{pmatrix}$$

$$= T_1 h^1 + T_2 h$$

$$\text{so} \lim_{h\to 0} \left| \frac{\sqrt{|h^1 h^2|} - \overset{\in\mathbb{R}}{T_1} h^1 - \overset{\in\mathbb{R}}{T_2} h^2}{|h|} \right| = 0$$

$T_1 h^1 + T_2 h^2 \longrightarrow \sqrt{|h^1 h^2|}$ with order of h as $h \longrightarrow 0$

for $h \to 0$ only on x axis i.e

$h^1 \to 0 \quad h^2 = 0$

then

$$\lim_{\substack{h^1 \to 0 \\ h^2 = 0}} \frac{|0 - T(h^1, 0)|}{|h^1|} = 0$$

$$\Rightarrow \lim_{h^1 \to 0} \frac{|T_1(\cancel{h^1})|}{|\cancel{h^1}|} = 0$$

$$\Rightarrow \underbrace{\lim_{h^1 \to 0} |T_1| = 0}_{\text{does not depend on } T_1}$$

so $T_1 = 0$

similarly $T_2 = 0$

now $T = [0 \quad 0]_{1 \times 2}$

but $$\lim_{h \to 0} \frac{\left|\sqrt{|h^1||h^2|} - 0\right|}{\sqrt{(h^1)^2 + (h^2)^2}}$$

for $h^1 = h^2 \to 0$ is

$$\lim_{h \to 0} \left| \frac{|\cancel{h^1}|}{\sqrt{2 |\cancel{h^1}|^{\cancel{2}}}} \right| = \frac{1}{\sqrt{2}} \neq 0$$

$\therefore$ not for all $h \to 0$

$$\lim_{h \to 0} \frac{|f(h) - f(0) - T(h)|}{|h|} = 0$$

$\therefore$ this is a contradiction

$\therefore$ No such T exist

$\therefore$ f is not diff at $(0,0)$

4. $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$
bilinear $x, x_1, x_2 \in \mathbb{R}^n$
$y, y_1, y_2 \in \mathbb{R}^m$

$f(ax_1, y) = af(x_1, y)$
$= f(x_1, ay)$

$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$
$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$

← doubt
what is
$\in \mathbb{R}^m$
$|(h,k)|$
$\in \mathbb{R}^n$ → how is this making sens

(i) To prove: given f is bilinear

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} = 0$$

proof: $h = \sum_{i=1}^{n} h^i e_i = \sum_{i=1}^{n} h^i \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$e^i \leftarrow$ Basis of $\mathbb{R}^n$

$k = \sum_{j=1}^{m} k^j \bar{e}_j = \sum_{j=1}^{m} k^j \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\bar{e}_j \leftarrow$ Basis of $\mathbb{R}^m$

$$|f(h,k)| = \left|f\left(\sum_{i=1}^{n} h^i e^i, k\right)\right|$$
$$= \left|\sum_{i=1}^{n} h^i f(e^i, k)\right|$$
$$= \left|\sum_{i=1}^{n}\sum_{j=1}^{m} h^i k^j f(e^i, \bar{e}^j)\right|$$
$$\leq \left|\sum_{i=1}^{n}\sum_{j=1}^{m} h^i k^j N\right| = \left|\sum_{i=1}^{n}\sum_{j=1}^{m} h^i k^j\right| |N| \leq |h||k||N|$$

cauchy ↙ swarts

$$N = \max\{|f(e_i, \bar{e}_j)| \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$$

where we are converting n or m to n, n or wlog $n > m$ and the new terms 0.

same for:

$$|(h,k)| = \sqrt{(h^1 - k^1)^2 + \cdots (h^m - k^m)^2 + (h^{m+1})^2 + \cdots (h^n)^2}$$
$$\geq |k| \quad \text{(our wlog case)}$$

$$\Rightarrow 1 \geq \frac{|k|}{|(h,k)|} \Rightarrow |h| \geq \frac{|k||h|}{|(h,k)|}$$

$\Rightarrow N|h| \geq \frac{N|k||h|}{|(h,k)|} \Rightarrow \lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} = 0$

(ii) <u>To prove</u>: If f is bilinear then $Df(a,b)(x,y) = f(a,y) + f(x,b)$

<u>proof</u>:

$$\lim_{(h,k)\to(0,0)} \frac{|f(a+h,b+k) - f(a,b) - f(a,k) - f(h,b)|}{|(h,k)|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\left| \cancel{f(a,b)} + \cancel{f(a,k)} + \cancel{f(h,b)} + f(h,k) - \cancel{f(a,b)} - \cancel{f(a,k)} - \cancel{f(h,b)} \right|}{|(h,k)|}$$

$= 0$ (already proved)

$\therefore \exists$ linear map $f(a,y) + f(x,b)$ s.t the above happens.

$f(a,y) + f(x,b)$ is linear from defination of f (trivial)

$\therefore Df(a,b)(x,y) = f(a,y) + f(x,b)$

Here we can also prove if

$|(a,b)| \leq 1$ then $|f(a,b)| \leq M$

and then $|(h,k)| < \varepsilon/M = \varphi$

then

$$\frac{|f(h,k)|}{|(h,k)|} = \frac{\varphi^{\cancel{2}} \left| f\left(\frac{h}{\varphi}, \frac{k}{\varphi}\right) \right|}{\cancel{\varphi}}$$

$$= |(h,k)| \left| f\left(\frac{h}{\varphi}, \frac{k}{\varphi}\right) \right|$$

$< \varepsilon$ as $\underbrace{\left| f\left(h/\varphi, k/\varphi\right) \right|}_{<1}$

5. $f:\mathbb{R}^n \to \mathbb{R}^n$ is diff

$f^{-1}:\mathbb{R}^n \to \mathbb{R}^n$ exist and is diff

<u>To prove</u>: $(f^{-1})'(a) = \left(f'(f^{-1}(a))\right)^{-1}$

<u>Proof</u>:

as $f \circ f^{-1}(x) = x \in \mathbb{R}^n$

$$D(f \circ f^{-1}(x)) = D(x) = D\begin{pmatrix} x^1 & & 0 \\ & x^2 & \\ 0 & & \ddots \; x^n \end{pmatrix} = I$$

$\Rightarrow$ By chain rule

$$D(f(f^{-1}(x))) \circ Df^{-1}(x) = I$$

$$\underbrace{\left(f(f^{-1}(a))\right)'}_{n\times n \text{ matrix}} \circ \underbrace{(f^{-1}(a))'}_{n\times n \text{ matrix}} = I$$

$$\Rightarrow (f^{-1}(a))' = \left[f(f^{-1}(a))\right]^{-1}$$

6. (a) $f(x,y,z) = \sin(x\sin y)$

$$D_1 f(x,y,z) = \partial/\partial x \, \sin(x\sin y)$$

$$= \cos(x\sin y)(\sin y)$$

$$D_2 f(x,y,z) = \partial/\partial y \, \sin(x\sin y)$$

$$= \cos(x\sin y) \times x\cos y$$

$$D_3 f(x,y,z) = \partial/\partial z \, \sin(x\sin y)$$

$$= 0$$

(b) $f(x,y) = \int_a^{x+y} g \quad g:\mathbb{R}\to\mathbb{R}$ is cont

$$D_1 f(x,y) = \partial/\partial x \int_a^{x+y} g(t)\,dt$$

$$= g(x+y)\left[\partial/\partial x(x+y)\right]$$

$$- g(a)\left[\partial/\partial x(a)\right]$$

$$= g(x+y)$$

Leibniz rule

$$f(x) = \int_{s(x)}^{g(x)} h(x,t)\,dt$$

$$df/dx = \int_{s(x)}^{g(x)} \partial_x h(x,t)\,dt$$

$$+ h(x, g(x)) \frac{dg}{dx}$$

$$- h(x, s(x)) \frac{ds}{d}$$

$$D_2 f(x, y) = g(x + y)$$

Quiz-1 :

Find the interior and boundry of $S = \{(x^1, x^2, \ldots x^n) \in \mathbb{R}^n \mid \text{each } x^i \in \mathbb{Q}\}$

Interior $(S) = \{x \in \mathbb{R}^n \mid \exists$ open retangle s.t $x \in A$, and $A \subseteq S\}$

Boundry $(S) = \{x \in \mathbb{R}^n \mid \forall B$ (open retangle) $\ni x$, $B \cap S \neq \phi$ and $B \cap [\mathbb{R}^n \setminus S] \neq \phi\}$

for $n = 1$

$S = \{x \in \mathbb{R} \mid x \in \mathbb{Q}\}$
$= \mathbb{Q}$

Because irrationales $(\mathbb{R} \setminus \mathbb{Q})$ are dense in $\mathbb{R}$, any open interval A will contain some irrational number Z.

A cannot be subset of S.

Interior $= \phi$

Boundry $= \mathbb{R}^n$

Tutorial-3 :

1. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$f(x) = f(x^1, x^2) = (e^{2x^1} + x^2,\ 3x^2 - \cos x^1,\ (x^1)^2 + x^2 + 2)$$

$$g(y) = g(y^1, y^2, y^3) = (3y^1 + 2y^2 + (y^3)^2,\ (y^1)^2 - y^3 + 1)$$

$F(x) = g \circ f(x)$

$DF(0) = D(g \circ f(0))$

$$= \underbrace{D(g(f(0)))}_{\substack{g:\ \mathbb{R}^3 \to \mathbb{R}^2 \\ 2\times 3 \text{ matrix}}} \circ \underbrace{D(f(0))}_{3\times 2 \text{ matrix}}$$

$g \circ f(x) : \mathbb{R}^2 \to \mathbb{R}^2$
2 x 2 matrix

$$D(g(y)) = \begin{bmatrix} D_1 g^1(y) & D_2 g^1(y) & D_3 g^1(y) \\ D_1 g^2(y) & D_2 g^2(y) & D_3 g^2(y) \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 3 & 2 & 2y^3 \\ 2y^1 & 0 & -1 \end{bmatrix}_{2\times 3}$$

all $D_i g^j(y)$ exist so g is diff

$$D(f(x)) = \begin{bmatrix} D_1 f^1(x) & D_2 f^1(x) \\ D_1 f^2(x) & D_2 f^2(x) \\ D_1 f^3(x) & D_2 f^3(x) \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 2e^{2x^1+x^2} & e^{2x^1+x^2} \\ \sin x^1 & 3 \\ 2x^1 & 1 \end{bmatrix}_{3\times 2}$$

as $D_i f^j(x)$ exist, f is diff

$f(0) = (e^0, -\cos(0), 2)$

$= (1, -1, 2)$

$$D(g(f(0))) = D(g(1,-1,2)) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix}_{2\times 3}$$

$$D(f(0)) = Df(0,0) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

$$D(g \circ f(0)) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

$$D(g \circ f(0)) = \begin{bmatrix} 6 & 13 \\ 4 & 1 \end{bmatrix}_{2\times 2}$$

now $D G(0) = D(f \circ g(4))$

$$= D(f(g(0))) \circ D(g(0))$$

$$= D(f(0,1)) \circ D(g(0))$$

$$= \begin{bmatrix} 2e & e \\ 0 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6e & 4e & -e \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}_{3\times 3}$$

2. $f : \mathbb{R}^n \to \mathbb{R}$ homogeneous of degree m

$f(tx) = t^m f(x) \quad \forall x \in \mathbb{R}^n$

and f is diff

so $D_i f^j(x)$ exist $\forall i, j = 1$
$\in \{1, n\}$

<u>To prove</u>: $\sum_{i=1}^{n} x^i D_i f(x) = m f(x)$

<u>proof</u>: let $g(t) = f(tx)$
then for $\phi(t) = tx$
putting x const
we get

$$g(t) = f \circ \phi(t)$$

where $\phi(t)$ is diff as it is a linear map (trivial)

and f given to be diff

$$\Rightarrow g'(t) = D(f \circ \phi(t))$$

$$= D(f(\phi(t))) \circ D(\phi(t))$$

$$= D(f(tx)) \circ D(tx)$$

$$= \underbrace{\begin{bmatrix} D_1 f(tx) \cdots\cdots D_n f(tx) \end{bmatrix}}_{1\times n} \cdot \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}}_{n\times 1} \quad — (1)$$

now also

$$g(t) = t^m f(x)$$

$$\Rightarrow g'(t) = m t^{m-1} f(x) \quad — (2)$$

putting $t=1$ in both we get

$$m f(x) = \begin{pmatrix} D_1 f(x) \cdots\cdots D_n f(x) \end{pmatrix} \circ \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

$$m f(x) = \sum_{i=1}^{n} x^i D_i f(x)$$

3. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$$

<u>To prove</u>: f is a C^∞ function and $f^{(i)}(0) = (0) \ \forall i$

<u>proof</u>: as $f: \mathbb{R} \to \mathbb{R}$

for $x \neq 0$ i.e $x \in \mathbb{R} \setminus \{0\}$

$$f'(x) = \frac{2}{x^3} e^{-1/x^2} = \frac{2}{x^3} f(x)$$

$$f''(x) = -\frac{6}{x^4} f(x) + \frac{2}{x^3} f'(x)$$

$$= -\frac{6}{x^4} f(x) + \frac{2}{x^3}\left(\frac{2}{x^3}\right) f(x)$$

$$f''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4}\right) f(x)$$

now if $f^{(k-1)}(x) = P_{3(k-1)}\left(\frac{1}{x}\right) f(x)$

$\hookrightarrow$ degree $3(k-1)$

then $$f^{k}(x) = \frac{\partial}{\partial x}\left(P_{3(k-1)}\left(\frac{1}{x}\right) f(x)\right)$$

$$= P'_{3(k-1)}\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) f(x) + P_{3(k-1)}\left(\frac{1}{x}\right)\left(\frac{2}{x^3} f(x)\right)$$

as P is diff degree will be added $\left(\frac{1}{x^n}\right)' = \left(\frac{-n}{x^{n+1}}\right)$

$$= Q_{3(k-1)+1}\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)f(x) + P_{3(k-1)}\left(\frac{1}{x}\right)\left(\frac{2}{x^3}f(x)\right)$$

$$= \gamma^1_{3(k-1)+1+2}\left(\frac{1}{x}\right)f(x)$$

$$+ \gamma^2_{3(k-1)+3}\left(\frac{1}{x}\right)f(x)$$

$$= \gamma_{3k}\left(\frac{1}{x}\right)f(x)$$

so by induction

$$f^{(k)}(x) = \underbrace{\gamma_{3k}\left(\frac{1}{x}\right)}_{\text{some polynomial of degree } 3k} f(x)$$

when $x \neq 0$

so $f \in C^\infty(\mathbb{R}\setminus\{0\})$

now for $x=0$

$$f^{(i)}(0) = \lim_{h\to 0} \frac{f^{(i-1)}(h) - f^{(i-1)}(0)}{h}$$

supposing for all $f^{(i-1)}(0)=0$

as $f^{(1)}(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$

$$= \lim_{h\to 0} \frac{\left(\frac{1}{h}\right)}{e^{\left(+\frac{1}{h^2}\right)}} \quad \frac{\infty}{\infty} \text{ form}$$

by l-Hospital

$$= \lim_{h\to 0} \frac{\left(+\frac{1}{h^2}\right)}{e^{\frac{1}{h^2}}\left(+\frac{2}{h^3}\right)} = \lim_{h\to 0} \frac{h}{2e^{1/h^2}} \to 0$$

as $h \to 0$

$\left(\frac{0}{\infty} \text{ form}\right)$

then $f^{(i)}(0) = \lim_{h\to 0} \frac{\gamma_{(3)(i-1)}\left(\frac{1}{h}\right)f(h) - 0}{h}$

$$= \lim_{h\to 0} \frac{\gamma_{(3)(i-1)}\left(\frac{1}{h}\right)f(h)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{\alpha_1}{h^{3(i-1)}} + \frac{\alpha_2}{h^{3(i-1)-1}} + \dots \alpha_k\right)f(h)}{h}$$

$$= \lim_{h \to 0} \left(\frac{\alpha_1}{h^{3(i-1)+1}} + \frac{\alpha_2}{h^{3(i-1)}} + \cdots \frac{\alpha_k}{h} \right) f(u)$$

$$= \lim_{h \to 0} \left(\frac{1}{h^{3(i-1)+1}} \right) \left(\alpha_1 + \alpha_2 h + \cdots \alpha_k h^{\gamma} \right) f(h)$$

so if we can show that

$\lim_{h \to 0} \frac{1}{h^{3(\gamma)+1}} f(h) \to 0$ we are done for $\gamma > 0$

as for $\gamma = 0$ this is proved

say true till $r = k-1$ (ie $\frac{f(u)}{h}, \frac{f(u)}{h^2}, \ldots \frac{f(u)}{h^{k-1}} \to 0$
then

$$\lim_{h \to 0} \frac{1}{h^{k-1}} f(u) = 0$$

now

$$\lim_{h \to 0} \frac{1}{h^k} f(u) \quad \frac{0}{0} \text{ form}$$

$$\Rightarrow \lim_{h \to 0} \frac{\left(\frac{1}{h^k}\right)}{\frac{1}{f(h)}} \quad \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{h \to 0} \frac{\left(\frac{+k}{h^{k+1}} \right)}{+\frac{1}{(f(u))^{\cancel{2}}} \times \left(\frac{2}{h^3} \cancel{f(u)} \right)}$$

$$= \lim_{h \to 0} \left(\frac{k}{h^{k+1}} \right) \times \frac{f(h)}{\frac{2}{h^3}}$$

$$= \lim_{h \to 0} \left(\frac{k}{2} \right) \frac{f(u)}{h^{k-2}}$$

$= 0$

as $\lim_{h \to 0} \frac{f(u)}{h^{k-2}} = 0$

for $k-2 = 1$ ✓

$k-2 = 2$

$\frac{f(u)}{h^2}$ lets check then

we are done

$$\lim_{h\to 0} \frac{f(h)}{h^2}$$

$$= \lim_{h\to 0} \frac{e^{-1/h^2}}{h^2}$$

let $\frac{1}{h^2} \to \infty$

$$= \lim_{n\to\infty} \frac{e^{-n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n \frac{1}{e^n}$$

by L-Hospitae

$$= \lim_{n\to\infty} \frac{1}{e^n} = 0$$

$\therefore$ By induction $f^{(i)}(0) = 0$

so $f \in C^\infty(\mathbb{R})$

4. $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

(a) $D_1 f((0,0))$ to exist

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$ should exist

$$= \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h}$$

$$= \lim_{h\to 0} 0 = 0$$

so yes the $D_1 f((0,0))$ exist

similary $D_2 f((0,0))$ exist

(b) for f to be diff at $(0,0)$

$\exists\, T: \mathbb{R}^2 \to \mathbb{R}$

s.t

$$\lim_{h\to 0} \frac{|f(h) - f(0) - T(h)|}{|h|} =$$

if $\exists$ such $T = [T_1 \quad T_2]_{1\times 2}$

then

$$\lim_{h\to 0} \frac{\left| f(h_1,h_2) - f(0,0) - [T_1 \quad T_2]\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right|}{|h|}$$

$$= \lim_{h\to 0} \frac{\left| \frac{h_1 h_2}{h_1^2+h_2^2} - 0 - T_1h_1 - T_2h_2 \right|}{\sqrt{h_1^2+h_2^2}}$$

this is true for all direction

for x-axis

$$= \lim_{h_1\to 0} \frac{|0-0 - T_1\cancel{h_1} - 0|}{|\cancel{h_1}|} = |T_1| = 0$$

$$\Rightarrow T_1 = 0$$

Similarly on y-axis $h_2 \to 0$, $h_1 = 0$

$$\Rightarrow T = [T_1 \quad T_2]_{1\times 2} = [0 \quad 0]_{1\times 2}$$

but for $h_1 = h_2$ we get:

$$\lim_{h_1\to 0} \frac{\left| \frac{h_1^2}{h_1^2+h_1^2} - 0 - 0 \right|}{\sqrt{h_1^2+h_1^2}}$$

$$= \lim_{h_1\to 0} \frac{\left|\frac{1}{2}\right|}{\sqrt{2}} \times \frac{1}{|h_1|}$$ this leads to infinity

$\therefore$ No such T exist

$\therefore$ not diff on $(0,0)$

(c) To check continuity we have to show

$$\lim_{h\to 0} f(h) = f(0)$$

here as $f(0) = f(0,0) = 0$

and $\lim_{h\to 0} f(h_1,h_2)$

$$= \lim_{h\to 0} \frac{h_1 h_2}{h_1^2+h_2^2}$$ but for $h_1 = h_2$

$$= \lim_{h_1\to 0} 1/2 \neq 0$$ so not true for all $h\to 0$

$\therefore$ f is not cont at $(0,0)$

5. $f(x,y) = \psi(ax+by)$

$a, b \in \mathbb{R}$

ψ is class C^q in some open set contaig (0)

$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$\psi(x) : \mathbb{R} \to \mathbb{R}$

$\underbrace{\psi(ax+by)}_{\mathbb{R}\to\mathbb{R}} = f(x,y)$

now as $\psi(ax+by) : \mathbb{R} \to \mathbb{R}$ taylor series can be applied to it

to get

$$\psi(\gamma_1) = \psi(\gamma_2) + \frac{\psi'(\gamma_2)(\gamma_2-\gamma_1)}{1!} + \frac{\psi^{(2)}(\gamma_2)(\gamma_2-\gamma_1)^2}{2!} + \dots + \frac{\psi^{(q-1)}(\gamma_2)(\gamma_2-\gamma_1)^{q-1}}{(q-1)!} + \frac{\psi^{(q)}(\gamma_2)(c)^q}{q!}$$

for some c b/w γ_1 and γ_2

then

$$\psi(ax+by) = f(x,y) = \frac{\psi(0)}{0!} + \frac{\psi'(0)(ax+by)^1}{1!} + \frac{\psi^{(2)}(0)(ax+by)^2}{2!} + \dots + \frac{\psi^{(q-1)}(0)(ax+by)^{q-1}}{(q-1)!} + \underbrace{\frac{\psi^{(q)}(0)(c)^q}{(q)!}}_{R_q(x,y)}$$

for some c b/w 0 and $ax+by$

$$f(x,y) = \sum_{m=0}^{q-1} \frac{\psi^{(m)}(0)}{m!} (ax+by)^m + R_q(x,y)$$

now $(ax+by)^m = \sum_{r=0}^{m} {}^mC_r (ax)^r (by)^{m-r}$

by binomial expansion

so, $$f(x,y) = \sum_{m=0}^{q-1} \frac{\psi^{(m)}(0)}{m!} \sum_{j=0}^{m} \binom{m}{j} (ax)^j (by)^{m-j} + R_q(x,y)$$

Assignment-1 :

Dhairya

Tutorial-4 :

1. $D_2 f = 0$

$$D_2 f = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = 0$$

now for y_1, y_2

$$\frac{f(y_1) - f(y_2)}{y_1 - y_2} = f'(y) \quad \text{for some } y \in (y_1, y_2)$$

then for $y_1 \neq y_2$

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y} = f'(x, y) = 0 \quad \text{for some } y \in (y_1, y_2)$$

$$\Rightarrow f(x, y_1) = f(x, y_2)$$

if $y_1 = y_2 \Rightarrow f(x, y_1) = f(x, y_1) = f(x, y_2)$ (trivial)

putting x const we get above

(b) $D_1 f = D_2 f = 0$ To show : f is const

proof : as $f(x, y_1) = f(x, y_2) \;\forall\; y_1, y_2 \in \mathbb{R}$

$f(x_1, y) = f(x_2, y) \;\forall\; x_1, x_2 \in \mathbb{R}$

putting $x = x_1$, $y = y_2$

$$\Rightarrow f(x_1, y_1) = f(x_1, y_2)$$
$$f(x_1, y_2) = f(x_2, y_2)$$

$$\Rightarrow f(x_1, y_1) = f(x_2, y_2) \;\forall\; x_1, y_1, x_2, y_2 \in \mathbb{R}$$

2. $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{\frac{-1}{(x-1)^2}} \cdot e^{\frac{-1}{(x+1)^2}} & ; x \in (-1,1) \\ 0 & ; x \notin (-1,1) \end{cases}$$

$g(x) = e^{\frac{-1}{(x-1)^2}} \to C^\infty \quad g'(1) = 0$

$h(x) = e^{\frac{-1}{(x+1)^2}} \to C^\infty \quad h'(-1) = 0$

from Q-3 by changing x linearly

now, $f(x) = \begin{cases} g(x) h(x) & ; x \in (-1,1) \\ 0 & ; x \notin (-1,1) \end{cases}$

$$1 = e^{-0} > e^{\frac{-1}{(x-1)^2}} > 0$$

$$1 = e^{-0} > e^{\frac{-1}{(x+1)^2}} > 0$$

so $0 < f(x) < 1$ for $x \in (-1,1)$

$\therefore$ positive

and given $f(x)=0$ for $x \notin (-1,1)$
$\therefore$ zero

now to show f is C^∞ : if $x \in (-1,1)$

$$f(x) = \underset{\substack{\uparrow \\ C^\infty \text{ for } x \in (-1,1)}}{g(x)}\, h(x) \leftarrow C^\infty \text{ for } x \in (-1,1)$$

$\Rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$

$f''(x) = g''(x)h(x) + 2g'(x)h'(x) + \cancel{g'(x)h'(x)} + g(x)h''(x)$

$f'''(x) = g'''(x)h(x) + 3g''(x)h'(x) + 3g'(x)h''(x) + g(x)h'''(x)$

1 1
1 2 1
1 3 3 1

$$f^{(n)}(x) = \sum_{i=0}^{n} {}^nC_i\, g^{(i)}(x)h^{(n-i)}(x)$$

as $g^{(i)}, h^{(n-i)}$ diff

$\Rightarrow f^{(n)}$ is diff

$\Rightarrow f$ is C^∞ for $x \in (-1,1)$

now for $x \notin (-1,1)$, for this lets work for $x \in (-\infty,-1) \cup (1,\infty)$
as
$f(x) = 0 \ \forall$ points
we have $f \in C^\infty$

now for $x = 1, -1$

wlog: at $x=1$ then

$$f'(x) = \lim_{h\to 0} \frac{f(1+h) - \overset{0}{f(1)}}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h)}{h} = \lim_{h\to 0} \frac{g(1+h)h(1+h)}{h}$$

for $h<0$

$$= \lim_{h\to 0^-} \frac{e^{-\frac{1}{(h)^2}}\, e^{-\frac{1}{(2)^2}}}{h}$$

in Q3-T3 we proved that this type is 0

so
$f'(1) = 0$
similarly $f'(-1) = 0$

and now $f^{(n)}(1) = \lim_{h\to 0^-} \dfrac{f^{(n-1)}(1+h) - 0}{h}$ } this is also 0 from binomial and Q-3 T-3 so $f \in C^{\infty}(\mathbb{R})$

$f > 0$ for $x \in (-1,1)$
and $f = 0$ for $x \notin (-1,1)$

3.(a) $f(x,y) = (e^x \cos y, e^x \sin y)$
now
if $f(x_1,y_1) = f(x_2,y_2)$

$\Rightarrow (e^{x_1}\cos y_1, e^{x_1}\sin y_1)$

$= (e^{x_2}\cos y_2, e^{x_2}\sin y_2)$

$e^{x_1-x_2}\cos y_1 = \cos y_2$ —— ①

$e^{x_1-x_2}\sin y_1 = \sin y_2$ —— ②

squaring and adding both term ①, ②

$e^{2x_1-2x_2} = 1$

$\Rightarrow e^{2x_1-2x_2} = e^0$

$\Rightarrow e^{2(x_1-x_2)} = e^0$

$\Rightarrow \log e^{2(x_1-x_2)} = \log e^0$

$\Rightarrow 2(x_1-x_2) = 0$

$\Rightarrow x_1 = x_2$

now $\sin y_1 = \sin y_2$
$\cos y_1 = \cos y_2$
$y_1, y_2 \in (0, 2\pi)$

for

$\sin(y_1-y_2) = \sin(y_1)\cos(y_2) - \cos(y_1)\sin(y_2)$
$= 0$
$\Rightarrow y_1 - y_2 = n\pi \ \forall n \in \mathbb{Z}$

$\cos(y_1-y_2) = \cos(y_1)\cos(y_2) + \sin(y_1)\sin(y_2)$
$= 1 \Rightarrow y_1 - y_2 = 2n'\pi \ \forall n' \in \mathbb{Z}$

as $y_1, y_2 \in (0, 2\pi)$

$y_1 - y_2 \in (-2\pi, 2\pi)$
$= 2n\pi \Rightarrow y_1 - y_2 = 0$
$\Rightarrow y_1 = y_2$

(b) $B = f(A)$

$= \left\{ f(x,y) \,\middle|\, \begin{matrix} y \in (0,2\pi) \\ x \in \mathbb{R} \end{matrix} \right\}$

$B = \left\{ (e^x \cos y, e^x \sin y) \,\middle|\, \begin{matrix} y \in (0,2\pi) \\ x \in \mathbb{R} \end{matrix} \right\}$

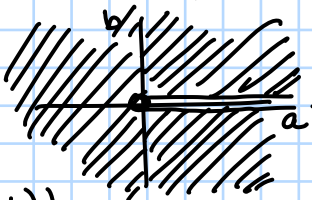
$(a,b) \in \mathbb{R}^2$
$e^x \cos y = a$
$e^x \sin y = b$
then $a^2 + b^2 = e^{2x}$
$\frac{1}{2}\log(a^2+b^2) = x$

also $\cos y = \frac{a}{\sqrt{a^2+b^2}}$ $\sin y = \frac{b}{\sqrt{a^2+b^2}}$

$a^2 < a^2 + b^2 \Rightarrow b^2 > 0$

if $a > 0$ then $b \neq 0$

$a < 0 \Rightarrow (a,b) \in (-\infty, 0] \cup (\mathbb{R})$, $(a,0) = f(x,y)$
$a > 0 \Rightarrow (a,b) \in (0,\infty) \cup (\mathbb{R} - \{0\})$ then



$\rightarrow \mathbb{R}\times\mathbb{R} - \{(a,0) \mid \forall a \geq 0\} = B$

$e^x \cos y = a$, $e^x \sin y = 0 \Rightarrow y = 0, \pi, 2\pi$

$\Rightarrow e^x(-1) = a$
$\Rightarrow e^x = -a$
if $a > 0$ in this case
$\Rightarrow e^x = -1$ *
$\therefore$ not possible

(c) $g'((0,1))$

$g = f^{-1}$

then $g f(x,y) = f^{-1} f(x,y)$
$g f(x,y) = (x,y)$

$$g'(f(x,y))\, f'(x,y) = I$$

$$g'(f(x,y)) = (f'(x,y))^{-1}$$

now

$f(x,y) = (0,1)$

for $e^x \cos y = 0$ $\quad y = \pi/2$ or $3\pi/2$ (not possible)
$e^x \sin y = 1$

$\Rightarrow e^x = 1$
$\Rightarrow x = 0$

so at $f(0, \pi/2) = (0,1)$

now, $g'(0,1) = (f'(0,\pi/2))^{-1}$

$$= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}^{-1}$$

$$g'(0,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is not-one one proof is

let f be one-one, wlog $D_1 f(x,y) \neq 0$

$\Rightarrow g(x,y) = (f(x,y), y)$

$\Rightarrow \det g'(x,y) = D_1 f(x,y) \neq 0$
and
$g(x,y) \in C^1$

$\Rightarrow$ on a line g is one-one

if $g^{-1}(u,v) = (b, y_0)$
$f(u,v) = b$
$v = y_0$

then $z \neq y_0$
$g(b,z) \neq g(b,y_0)$
but
$f(x,y) = b \Rightarrow y = y_0 \Rightarrow z = y_0$ *

$\therefore f$ is not-one-one

4. $f: \mathbb{R}^2 \to \mathbb{R}^2$
$f(x,y) = (xy, x^2+y^2+e^{(x-2)(y-1)})$

<u>To prove</u> : $\exists r>0$ s.t $\forall (a,b) \in B_r((2,6))$
$\exists (x,y) \in \mathbb{R}^2$ s.t $f(x,y) = (a,b)$

<u>proof</u> : $f'(x,y) = \begin{bmatrix} y & x \\ 2x + e^{(x-2)(y-1)}(y-1) & 2y + e^{(x-2)(y-1)}(x-2) \end{bmatrix}$

also as $D_i f_j$ exist and is cont $\forall i,j \Rightarrow f \in C'$

now $x=2$ $y=1$ then

$f(x,y) = f(2,1) = (2, 4+1+e^0) = (2,6)$

$\det f'(2,1) = \det \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \neq 0$

By inverse function theorem, $\exists$ W ← open in $\mathbb{R}^2$
in neighborhood of $(1,2)$
s.t
$f: W \to V$ is invertible
(V ← open in $\mathbb{R}^2$)

and as V is open, as $(1,2) \in W \Rightarrow f((1,2)) = (2,6) \in V$
$\exists r > 0$ s.t
$B_r((2,6)) \subseteq V$

then
$f^{-1}(B_r((2,6))) \subseteq W$
as $f^{-1}: V \to W$

so, $\forall (a,b) \in B_r((2,6))$
$\exists (x,y) \in W \subseteq \mathbb{R}^2$ s.t
$f^{-1}((a,b)) = (x,y)$
$\Rightarrow (a,b) = f(x,y)$

5. $f: \mathbb{R} \to \mathbb{R}$
$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

(a) <u>To prove</u> : f is diff

<u>proof</u> : for $x \neq 0$ $\frac{\partial}{\partial x}\left(x + 2x^2 \sin\left(\frac{1}{x}\right)\right)$

$= 1 + 4x\sin\left(\frac{1}{x}\right) + 2\cancel{x^2}\cos\left(\frac{1}{x}\right)\left(\frac{-1}{\cancel{x^2}}\right)$

$= 1 + 4x\sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right)$

for $x \neq 0$ f is diff

for $x = 0$
$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \frac{\cancel{h} + 2h^{\cancel{2}}\sin\left(\frac{1}{h}\right)}{\cancel{h}} = 1$

then f is diff at all the points

but f is not C^1 at all points
as

$f'(0)=1$

$\lim_{h\to 0} f'(h) = \lim_{h\to 0}\left(1+4h\sin\left(\frac{1}{h}\right)-2\cos\left(\frac{1}{h}\right)\right)$

is not defined as $\cos\left(\frac{1}{h}\right)$ is not defined

$\therefore f$ is not $C^1(\mathbb{R})$

$$f'(x) = \begin{cases} 1+4x\sin\left(\frac{1}{x}\right)-2\cos\left(\frac{1}{x}\right) & ; x\neq 0 \\ 1 & ; x=0 \end{cases}$$

(b) <u>To prove</u> : $\forall n\in\mathbb{N}_+$ $\quad I_n = \left(\frac{2}{(4n+1)\pi}, \frac{1}{2n\pi}\right)$

contains x_n s.t

f is local max at x_n

<u>proof</u> : $f'\Big(\underbrace{\frac{2}{(4n+1)\pi}}_{x_0}\Big) = 1+4\left(\frac{2}{(4n+1)\pi}\right)1-2(0)$

$= 1+\frac{8}{(4n+1)\pi}$

$f'\Big(\underbrace{\frac{1}{2\pi n}}_{x_1}\Big) = 1+4\left(\frac{1}{2\pi n}\right)(0)-2(1) = -1$

$= -1$

as $n\in\mathbb{N}_+$ $\quad f'(x_0)>0 \ \forall n\in\mathbb{N}_+$

$f'(x_1) = -1<0 \ \forall n\in\mathbb{N}_+$

and as for $x\neq 0$ $f\in C^1(\mathbb{R}\setminus\{0\})$

By IVT, $\exists x_n$ s.t

$f'(x_n)=0$ now as

for $x\neq 0$,

$f''(x) = \frac{d}{dx}\left(1+4x\sin\left(\frac{1}{x}\right)-2\cos\left(\frac{1}{x}\right)\right)$

$= 4\sin\left(\frac{1}{x}\right)+4x\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)+2\sin\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$

$f''(x) = 4\sin\left(\frac{1}{x}\right)-\frac{4}{x}\cos\left(\frac{1}{x}\right)-\frac{2}{x^2}\sin\left(\frac{1}{x}\right)$

$= \left(4-\frac{2}{x^2}\right)\sin\left(\frac{1}{x}\right)-\frac{4}{x}\cos\left(\frac{1}{x}\right)$

$\frac{1}{x}\in\left(2n\pi, 2n\pi+\pi/2\right)$

for $n=1$ $\quad 1/x\in(2\pi, 2\pi+\pi/2)$

$\frac{1}{x^2} \in (4\pi^2, (2.5)^2\pi^2)$

$\frac{1}{x^2} = (4\pi^2, 6.25\pi^2)$

$-\frac{1}{x^2} = (-6.25\pi^2, -4\pi^2)$

$4 - \frac{1}{x^2} = (4 - 6.25\pi^2, 4 - 4\pi^2) < 0$

$-\frac{4}{x} = (-10\pi, -8\pi) < 0$

so $f''(x) < 0$ for all x in that interval
same for all n

$\Rightarrow$ as $f'(x_n) = 0$ and $f''(x_n) < 0$

$\Rightarrow x_n \leftarrow$ is a point of maxima

(c) <u>To prove</u>: every open Interval I containing 0, $\exists$ distinct points $y_1, y_2 \in I$ s.t $f(y_1) = f(y_2)$

<u>proof</u>: any open interval contg 0
$\exists r > 0$ s.t

$B_r(0) \subseteq I$

now let this Be $(-r, r)$

then for $(0, r) \subseteq I$

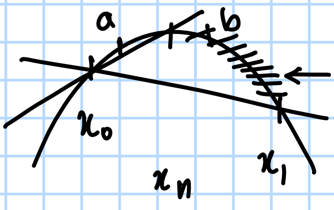
now let $n = \left\lfloor \frac{1}{2\pi r} \right\rfloor$

then $\frac{1}{2\pi r} + 1 > n > \frac{1}{2\pi r}$

then $\exists n \in \mathbb{N}_+$ s.t

$\left(\frac{2}{(4\pi n + 1)}, \frac{1}{2\pi n}\right) \subseteq I$
$\uparrow$
neighbourhood of 0

so, $\exists x_n \in \left(\frac{2}{4\pi n + 1}, \frac{1}{2\pi n}\right)$ s.t



$f''(x_n) < 0, \quad f'(x_n) = 0$

now if
$f(a) \leq f(b)$
$a \in (x_0, x_n)$
$b \in (x_n, x_1)$
then $f(a), f(b) \leq f(x_n)$
$\downarrow$
maxima

By IVT $\exists c \in (a, x_n)$
s.t $f(c) = f(b)$ $\therefore$ not one-one

Tutorial-5 :

1. $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f(x,y) = x^2 - y^2$

(a) $f'(x,y) = \begin{bmatrix} D_1 & D_2 \end{bmatrix}_{1\times 2}$

$f'(x,y) = \begin{bmatrix} 2x & -2y \end{bmatrix}_{1\times 2}$

(b) $I = (-a, a)$

for $y = g(x)$

$f(x, g(x)) = 0$

$x^2 - y^2 = 0 \Rightarrow y = \pm x$

for $g: I \longrightarrow \mathbb{R}$

$x \longmapsto x$

we have

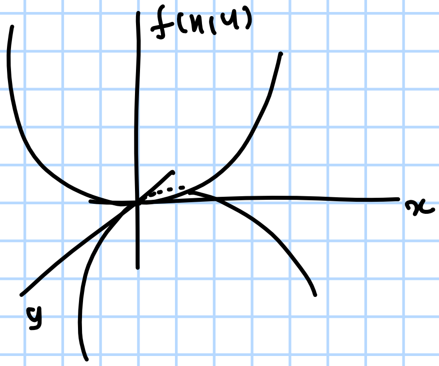
$f(x, g(x)) = x^2 - (x)^2 = 0$

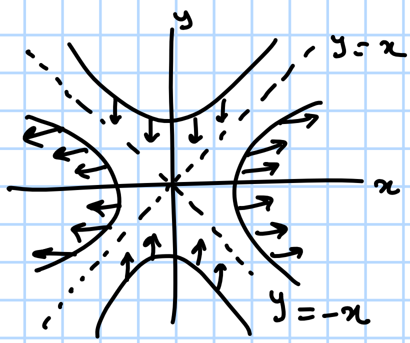
$\forall x \in (-a, a) = I$

(c) conclusion of implicit theorm fails as for I, there is not a unique $g(x)$ s.t $f(x, g(x)) = 0$

$g(x) = x$ or $-x$

$\therefore$ $g(x)$ is ~~not~~ unique (this is because $f(0,0) = 0$ but $\det(M) = 0$)





2. $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ class c^1

$f(x, y^1, y^2) = f(x, y)$

$x \in \mathbb{R}$

$y \in \mathbb{R}^2$

$f: \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

given $f(3,-1,2) = 0$

$f'(3,-1,2) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{2\times 3}$

(a) To prove: $\exists\, g : A \to \mathbb{R}^2$ of C^1 s.t $A \subseteq \mathbb{R}$ open, $f(x, g^1(x), g^2(x)) = 0$ $\forall x \in A$

$g(3) = (-1, 2)$

proof:

now as $f(3, -1, 2) = 0$

for $x = 3 = a$

$y = (-1, 2) = b$

$$M_{2\times 2} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$\det(M_{2\times 2}) = 2 + 1 = 3 \neq 0$, so

as f is C^1 $\Rightarrow$ cont and diff on any open set cont (a, b)

and $\det(M_{2\times 2}) \neq 0$

for $f(a, b) = 0$

then by Implicit fution theorm:

$\exists\, A \subseteq \mathbb{R}$ open and $B \subseteq \mathbb{R}^2$ open s.t $a \in A$, $b \in B$

and $\forall x \in A$, $\exists$ unique $g(x) \in B$

s.t $f(x, g(x)) = 0$

g is diff

now, as $a = 3$, $\exists\, A \ni 3$ s.t $A \subseteq \mathbb{R}$

and $g : A \to B \subseteq \mathbb{R}^2$

s.t $\forall x \in A$, $\exists\, g(x) \in B$

and $g(3) = b = (-1, 2)$

so $g(3) = (-1, 2)$

as g is diff

now as $\underbrace{\frac{\partial}{\partial x} f(x, g(x))}_{2\times 1 \text{ matrix}} + \frac{\partial}{\partial y} \underbrace{f(x, g(x))}_{2\times 2 \text{ matrix}} \circ \underbrace{g'(x)}_{2\times 1 \text{ matrix}} = 0$

$$g'(x) = \left[\frac{\partial}{\partial y} f(x, g(x))\right]^{-1} \frac{\partial}{\partial x} f(x, g(x))$$

in a small open set cont (a, b) this is not zero as $\det(M) \neq 0$ and f is C^1

as all entries of $\frac{\partial}{\partial x} f(x, g(x))$ is cont and so is $\frac{\partial f}{\partial y} f(x, g(x))$ we have

$g'(x) \to$ cont on small interval cont (a, b)

$\Rightarrow g$ is C^1 for $g : A \to B$

(b) $\phi : A \to \mathbb{R}^2$

$\phi(x) = f(x, g(x))$

$D\phi(x) = D(f(x, g(x)))$

let

$\varphi(x) = (x, g(x))$

then

$D\phi(x) = D(f(\varphi(x)))$

by chain rule

$$D\phi(x) = Df(\varphi(x)) \circ D(\varphi(x))$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} f(x, g(x)) & \frac{\partial}{\partial y} f(x, g(x)) \end{bmatrix} \circ \begin{bmatrix} \frac{\partial}{\partial x} x \\ Dg(x) \end{bmatrix}$$

$$D\phi(x) = \frac{\partial}{\partial x} f(x, g(x)) + \left(\frac{\partial}{\partial y} f(x, g(x))\right)\left(Dg(x)\right)$$

now as for $\forall x \in A$
$f(x, g(x)) = 0$

$$\Rightarrow D\phi(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left(\text{const function diff}\right)$$

now

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \circ 1 + \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} Dg(3)$$

$$\Rightarrow \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} Dg(3)$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \frac{1}{3}$$

as

$$\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = Dg(3)$$

$$\Rightarrow Dg(3) = \frac{1}{3}\begin{bmatrix} -1+1 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow Dg(3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

3. $f: \mathbb{R}^5 \longrightarrow \mathbb{R}^2$ of C^1

$P = (1, 2, -1, 3, 0)$

$$f'(P) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}$$

$f(P) = 0$

(a) <u>To prove</u>: $\exists g: A \to \mathbb{R}^2$ of C^1 on A open of $\mathbb{R}^3$ s.t

$f(x^1, g^1(x), g^2(x), x^2, x^3) = 0$ for $x = (x^1, x^2, x^3)$
$g(1, 3, 0) = (2, -1)$

proof: now

$f(x^1, g^1(x), g^2(x), x^2, x^3)$

$$= f(h(x^1, x^2, x^3, g^1(x), g^2(x)))$$

for $h(x^1, x^2, x^3, g^1(x), g^2(x))$

$$= (x^1, g^1(x), g^2(x), x^2, x^3)$$

let $\phi(x,y) = f(h(x,y))$

now $\phi'(x,y) = \underbrace{f'(h(x,y))}_{\text{given}} \circ \underbrace{h'(x,y)}_{\text{to find}}$

$$= f'(h(x,y)) \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{5\times 5}$$

now $\phi((1,3,0),(2,-1)) = f(P) = 0$ —— ①

as $\phi(x,y) = f(h(x,y))$
where
h is a linear map
and f is C^1 $\Rightarrow \phi(x,y)$ is C^1 —— ②

$$\phi'((1,3,0),(2,-1)) = f'(P) \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{5\times 5}$$

$$= \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\phi'(a,b) = \begin{bmatrix} 1 & -1 & 2 & 3 & 1 \\ 0 & 2 & -4 & 0 & 1 \end{bmatrix} \text{ —— ③}$$

where $a = (1,3,0)$
$b = (2,-1)$

now on $\phi(x,y)$ when we apply implicit funtion theorm
① ϕ is C^1
② $\phi(a,b) = 0$
③ $\phi'(a,b)$ is s.t
the $M_{2\times 2} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$ has $\det(M_{2\times 2}) = 3 \neq 0$

then $\exists$ A open $\subseteq \mathbb{R}^3$ s.t $a \in A$
B open $\subseteq \mathbb{R}^2$ s.t $b \in B$
unique $g: A \to \mathbb{R}^2$ where $\phi(x, g(x)) = 0$ $\forall x \in A$

$g(a)=b \Rightarrow g(1,3,0)=(2,-1)$

also from

$$\frac{\partial}{\partial x}\phi(x,g(x)) + \frac{\partial}{\partial y}\phi(x,g(x))\circ g'(x)=0$$

as g is diff $\Rightarrow g$ is C^1 on $A \cap$ small set where $\frac{\partial}{\partial y}\phi(x,g(x))$ is invertible

let new $A = A \cap$ small set

(b) $\phi(x,g(x))=0 \quad \forall x \in A$

$$\Rightarrow D\phi(x,g(x))=0=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$$

now $D\phi(x,g(x)) = \frac{\partial}{\partial x}\phi(x,g(x))I + \frac{\partial}{\partial y}\phi(x,g(x))\circ D(g(x))$

for $x=a$
$g(x)=b$

we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix}_{2\times 3} + \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}_{2\times 2}\circ D(g(a))$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}_{2\times 2}\circ D(g(a))$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} = D(g(a))$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} -1 & 3 & -6 \\ 0 & -6 & 12 \end{bmatrix}_{2\times 3} = \begin{bmatrix} -1/3 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}_{2\times 3} = D(g(a))$$

4. $3x+y-z+u^2=0$
$x-y+2z+v=0$
$2x+2y-3z+2u=0$

$$f(x,y,z,u) = x\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + u\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$f'(x,y,z,u) = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}_{3\times 4}$$

now if x,y,u in terms of z can solve this i.e

$\phi(z,x,y,u) = f(h(z,x,y,u))$
where
$h(z,x,y,u) = (x,y,z,u)$

$$h'(z,x,y,u) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

now for $\phi(z, x, y, u)$

if we have

$g(z) = (x, y, u)$

then we are done

now

$$\phi(z, x, y, u) = x\begin{pmatrix}3\\1\\2\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}-1\\2\\-3\end{pmatrix} + u\begin{pmatrix}u\\1\\2\end{pmatrix}$$

$$\phi'(z, x, y, u) = \begin{bmatrix}-1 & 3 & 1 & 2u\\ 2 & 1 & -1 & 1\\ -3 & 2 & 2 & 2\end{bmatrix}$$

now for $\phi(a, b) = 0$ (zeroes of the function)

i.e $a = z$

$b = (x, y, u)$

or solution

then $M_{3\times3} = \begin{bmatrix}3 & 1 & 2u\\ 1 & -1 & 1\\ 2 & 2 & 2\end{bmatrix}$

$\det M_{3\times3} = -6 + 4u + \cancel{2} + 4u - 6 - \cancel{2}$

$= 4(2u-3)$

as $u \neq 3/2$

$\det M_{3\times3} \neq 0$ for (a, b)

and by implicit function theorm

can be solved for x, y, u in terms of z

similary for y, z, u in terms of x

as now the

$M_{3\times3} = \begin{bmatrix}1 & -1 & 2u\\ -1 & 2 & 1\\ 2 & -3 & 2\end{bmatrix}$

$\det M_{3\times3} = \cancel{4} + 6u - \cancel{2} - 8u + 3 - \cancel{2}$

$= 3 - 2u$

as $u \neq 3/2$

$\det M_{3\times3} \neq 0$

but for x, y, z in terms of u:

$M_{3\times3} = \begin{bmatrix}3 & 1 & -1\\ 1 & -1 & 2\\ 2 & 2 & -3\end{bmatrix}_{3\times3}$

$\det M_{3\times3} = +\cancel{9} - \cancel{2} \cancel{+4} \cancel{-2} - \cancel{12} \cancel{+3}$

$= 0$

$$\begin{cases} 3x + y - z = -u^2 \\ x - y + 2z = -u \\ 2x + 2y - 3z = -2u \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 3 & 1 & -1 & -u^2\\ 1 & -1 & 2 & -u\\ 2 & 2 & -3 & -2u \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & -u\\ 0 & 4 & -7 & -u^2+3u\\ 0 & 0 & 0 & u^2-3u \end{array}\right)$$ if $f(a, b) = 0$

then $u = 0, 3 \leftarrow$ inf many else no

so if u not 0 or 3
then no solutions

if u 0 or 3 $\Rightarrow$ no unique solutions

5. To prove : $\exists\, g: \mathbb{R} \to [0,1]$ s.t
$g \in C^\infty$

$g(x) = 0 \;\; \forall x \leq 0$
$g(x) = 1 \;\; \forall x \geq \varepsilon$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} e^{-\frac{1}{(x-\varepsilon)^2}} & ; x \in (0,\varepsilon) \\ 0 & ; x \in \mathbb{R} \setminus (0,\varepsilon) \end{cases}$$

now f is C^∞ (proved already)

$$g(x) = \frac{\int_0^x f(t)\,dt}{\int_0^\varepsilon f(t)\,dt}$$

for $x \leq 0$

$$f(x) = 0$$
$$\Rightarrow g(x) = \frac{\int_0^x (0)\,dt}{\int_0^\varepsilon f(t)\,dt} = 0$$

now for $x \geq \varepsilon$:

$$g(x) = \frac{\int_0^x f(t)\,dt}{\int_0^\varepsilon f(t)\,dt} = \frac{\int_0^\varepsilon f(t)\,dt}{\int_0^\varepsilon f(t)\,dt} + \frac{\int_\varepsilon^x f(t)\,dt}{\int_0^\varepsilon f(t)\,dt}$$

this is 0 by construction

$$= 1$$

$$\forall x \geq \varepsilon$$

as $f > 0$ for $x \in (0,\varepsilon)$

$$\Rightarrow \int_0^\varepsilon f(t)\,dt > 0$$

and now

$$g(x) = \frac{\int_0^x f}{\int_0^\varepsilon f}$$

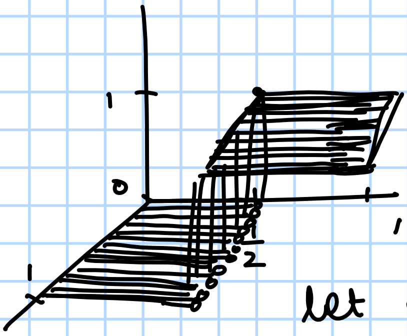
$$g'(x) = \frac{f(x)}{\int_0^\varepsilon f}$$

$\Rightarrow g'$ is C^∞ as f is C^∞
$\Rightarrow g$ is C^∞ as $g' = f \leftarrow$ cont and C^∞

Tutorial-6 :

1. $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 0 & ; \ 0 \leqslant x < 1/2 \\ 1 & ; \ \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$



let the partition be:

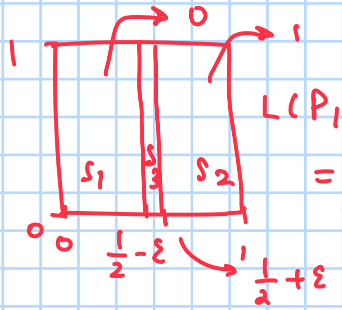
on x-axis : $[0,1]$ as:

$$P_1 = \left\{ 0, \frac{1}{2n+1}, \frac{2}{2n+1}, \ldots\ldots \frac{2n+1}{2n+1} \right\}$$

on $[0,1]$ y-axis:

$$P_2 = \{0, 1\}$$

then partition is $P = (P_1, P_2)$



$$L(P,f) = \sum m_S(f)\, v(S) = 0 \times v(S_1) + 0 \times v(S_3) + 1 \times \left(\frac{1}{2} - \varepsilon\right)$$

$$U(P,f) = 0 \times v(S_1) + 1 \times (2\varepsilon) + 1 \times \left(\frac{1}{2} - \varepsilon\right)$$

$$U(P,f) - L(P,f) = 2\varepsilon$$

now, every closed rectangle is of form

$$\left[\frac{i}{2n+1}, \frac{i+1}{2n+1}\right] \times [0,1]$$

now let $S_i = \left[\frac{i-1}{2n+1}, \frac{i}{2n+1}\right] \times [0,1]$

and for $i = 1, 2, \ldots\ldots n$

we have

$m_{S_i}(f) = 0$ and $M_{S_i}(f) = 0$

as $f \equiv 0$ on this interval

and similarly for $i = n+2, n+3 \ldots 2n+1$

$f \equiv 1 \Rightarrow m_{S_i}(f) = 1$

$M_{S_i}(f) = 1$

$$\frac{n}{2n+1} < \frac{1}{2} = \frac{n}{2n} < \frac{n+1}{2n+1}$$

we have

$\frac{1}{2}$ in $[0,1]$ to be

s.t

$$\frac{1}{2} \in \left[\frac{n}{2n+1}, \frac{n+1}{2n+1}\right]$$

so for S_{n+1}

$$m_{S_{n+1}}(f) = 0$$

$$M_{S_{n+1}}(f) = 1$$

and now $L(f, P_n) = \sum_{i=1}^{2n+1} m_{S_i}(f)\, v(S_i) = \dfrac{n}{2n+1}$

as $v(S_i) = (1)\left(\frac{1}{2n+1}\right)$

$$U(f,P_n) = \sum_{i=1}^{2n+1} M_{S_i}(f)\, v(S_i) = \frac{n+1}{2n+1}$$

$$U(f,P_n) - L(f,P_n) = \frac{1}{2n+1}$$

make n small enough s.t

$$\frac{1}{2n+1} < \varepsilon$$

so, $\forall \varepsilon > 0, \exists P \in \mathcal{P}([0,1]\times[0,1])$ s.t

$$U(f,P_n) - L(f,P_n) < \varepsilon$$

$\Rightarrow f$ is integrable

now $\forall P \in \mathcal{P}([0,1]\times[0,1])$

$$L(f,P) \leqslant U(f,P_n)$$

as $n \to \infty$

$$\Rightarrow L(f,P) \leqslant \frac{1}{2}$$

now similarly $U(f,P) \geqslant L(f,P_n)$ as $n \to \infty$

$$\Rightarrow U(f,P) \geqslant \frac{1}{2} \quad \forall P \in \mathcal{P}([0,1]\times[0,1])$$

so $\quad L(f,P) \leqslant \frac{1}{2} \leqslant U(f,P')$

for $P, P' \in \mathcal{P}[0,1]\times[0,1]$

as f is integrable $\Rightarrow \overline{\lim_{P\in\mathcal{P}}} L(f,P) = \underline{\lim_{P\in\mathcal{P}}} U(f,P)$

as $\overline{\lim_{P\in\mathcal{P}}} L(f,P) \leqslant \frac{1}{2}$

$\underline{\lim_{P\in\mathcal{P}}} U(f,P) \geqslant \frac{1}{2} \quad \Rightarrow \overline{\lim_{P\in\mathcal{P}}} L(f,P) = \frac{1}{2}$

$$= \underline{\lim_{P\in\mathcal{P}}} U(f,P)$$

$$\Rightarrow \int_{[0,1]\times[0,1]} f = \frac{1}{2}$$

2. $f : [0,1]\times[0,1] \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} 0 \ ; & x \neq y \\ 1 \ ; & x = y \end{cases}$$

now for (x,y) s.t $x \neq y$
f is cont
as $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$$f(B((x,y),\delta)) \subseteq B(f(x,y),\varepsilon)$$

this is trivial as $f \equiv 0$ on this ball
for small enough δ

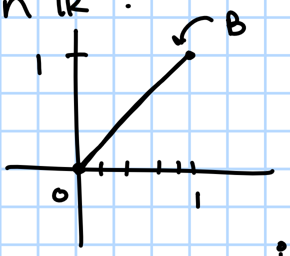
also for $x_0 = y_0$ f is not continious as:

for $B_\delta(x_0, y_0)$
there are points
s.t $x \neq y \Rightarrow f = 0$
and for $x_0 = y \Rightarrow f = 1$
inside this ball
for $\varepsilon < 1$ <u>not</u> possible

$\therefore$ $B = \{(x,y) \mid f \text{ is } \underline{\text{not}} \text{ cont on } (x,y)\}$

$= \{(x,y) \mid x = y, \quad x,y \in [0,1]\}$

now in $\mathbb{R}^2$:



the measure of $B = 0$
as:
B can be covered using
$\bigcup_{i=1}^{n} \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$
where $U_i = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$

and $\sum_{i=1}^{n} v(U_i) = \sum_{i=1}^{n} \left(\frac{1}{n^2}\right) = \frac{1}{n} < \varepsilon$
by archemedian
property

then, measure of $B = 0$
$\Rightarrow f$ is integrable

3. $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$
has measure 0.

$\longrightarrow \underbrace{[-i, i] \times [-i \times i] \times \cdots [-i \times i]}_{n-1 \text{ times}} \times \{0\}$ is a closed rectangle

now $\bigcup_{i=1}^{\infty} \underbrace{[-i,i] \times [-i,i] \times \cdots \times [-i \times i]}_{n-1 \text{ times}} \times \{0\} = \mathbb{R}^{n-1} \times \{0\}$

so it covers $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$

now $\sum_{i=1}^{\infty} v(U_i) = \sum_{i=1}^{\infty} (0)$
$\longrightarrow$ vol $= 0$ as

$\text{vol}(U_i) = (2i)^{n-1} \times (0) = 0$

$\Rightarrow \sum v(U_i) = 0 < \varepsilon$

$\therefore$ measure of $\mathbb{R}^{n-1} = 0$ in $\mathbb{R}^n$

4. $f : [a,b] \longrightarrow \mathbb{R}$
↳ increasing function

To prove: $\{x \mid f$ is discont at $x\}$ has measure 0

proof:

we know that $O(f,x) = 0 \Leftrightarrow f$ is cont at $x = 0$

now so f is discont at x then

$O(f,x) > 0$

now let $B = \{x \in [a,b] \mid f$ is discont at $x\}$

$= \{x \in [a,b] \mid O(f,x) > 0\}$

let $B_{\frac{1}{n}} = \{x \in [a,b] \mid O(f,x) > \frac{1}{n}\}$

for $n \in \mathbb{N}$

then $B = B_1 \cup B_{\frac{1}{2}} \cup B_{\frac{1}{3}} \ldots$

now $B_{\frac{1}{n}} = \{x \in [a,b] \mid O(f,x) > \frac{1}{n}\}$

as $O(f,x) = \lim_{\delta \to 0} M(x,f,\delta) - m(x,f,\delta)$

if $O(f,x) > \frac{1}{n}$ then

$\sum_{x \in B_{\frac{1}{n}}} O(x,f) < f(b) - f(a)$

(from tutorial-1, Q-5)

as f is inc function

now as $\sum_{x \in B_{\frac{1}{n}}} O(x,f) > \sum_{x \in B_{\frac{1}{n}}} \left(\frac{1}{n}\right)$

but $\sum_{x \in B_{\frac{1}{n}}} O(x,f) < f(b) - f(a)$

$\Rightarrow \sum_{x \in B_{\frac{1}{n}}} \left(\frac{1}{n}\right) < f(b) - f(a)$

here $|B_{\frac{1}{n}}| < n(f(b) - f(a))$

$\Rightarrow |B_{\frac{1}{n}}|$ is finite

$\forall n \in \mathbb{N}$

so $B_{\frac{1}{n}}$ has finite x

then says

$$\{x_1, x_2, \ldots x_r\} = B_{\frac{1}{n}}$$

then measure of $B_{\frac{1}{n}} = 0$ as

we can take

$$U_i = \left[x_i - \frac{\varepsilon}{3r}, x_i + \frac{\varepsilon}{3r}\right]$$

s.t $\sum_{i=1}^{r} v(U_i) = \sum_{i=1}^{r} \left(\frac{2\varepsilon}{3r}\right) < \varepsilon$

so measure of $B_{\frac{1}{n}} = 0 \;\forall n \in \mathbb{N}$

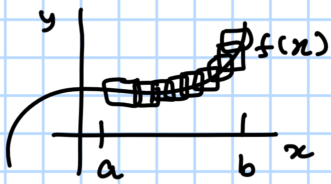
$\Rightarrow$ measure of $B \left(B_1 \cup B_{\frac{1}{2}} \cup \ldots \right) = 0$

5. $f : [a,b] \longrightarrow \mathbb{R}$

graph of f is subset $G_f = \{(x,y) \mid y = f(x)\} \subseteq \mathbb{R}^2$

<u>To prove</u> : f is cont $\Rightarrow G_f$ has measure 0 in $\mathbb{R}^2$

<u>proof</u> :



as $f : [a,b] \rightarrow \mathbb{R}$ $\hookrightarrow$ Bounded

and as f is cont on $[a,b]$

$\Rightarrow f$ is uniformly cont on $[a,b]$

or $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\forall x,y \in [a,b]$

where

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

now

$[a,b]$ can be split into :

$$\{a, a+\delta, a+2\delta, \ldots a+(n-1)\delta, b\}$$

where $n\delta < b-a$
$(n+1)\delta \geq b-a$

i.e

$(n)\delta < (b-a) \leq (n+1)\delta$

then $(n\delta) + a < b$

$a < a+\delta < a+2\delta < \cdots < a+n\delta < b$

now then in the interval :

$$[i\delta + a, (i+1)\delta + a] = P_i$$

for $i < n$ we have

$\forall x, y \in P_i \Rightarrow |x-y| < |(i+1)\delta + a - i\delta - a|$
$= \delta$

$\Rightarrow |x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

let $P_i' = [m_{P_i}(f), M_{P_i}(f)]$

then $|M_{P_i}(f) - m_{P_i}(f)| < \varepsilon$ from above

and
$P_i'' = P_i \times P_i'$
$= [i\delta + a, (i+1)\delta + a] \times [M_{P_i}(f) - m_{P_i}(f)]$

this covers the graph in P_i'' as:
$\forall x \in P_i$
$m_{P_i}(f) \leq f(x) \leq M_{P_i}(f)$

for $i = n$ $[n\delta + a, b] = P_n$
$\forall x, y \in P_n$
we have
$|x - y| < \delta$
and
so $|M_{P_n} - m_{P_n}| < \varepsilon$
similar calculation

and so
$$\left(\bigcup_{i=1}^{n-1} P_i''\right) \cup \left(P_n''\right) \text{ covers } G_f \text{ in } \mathbb{R}^2$$

for $i < n$
now $v(P_i'') = v\left([i\delta + a, (i+1)\delta + a] \times [M_{P_i} - m_{P_i}]\right)$
$< (\delta)(\varepsilon)$

$v(P_n'')$ is also $< (\delta)(\varepsilon)$

then $\sum_{i=1}^{n} v(P_i'') < n \times \delta \times \varepsilon < (b-a)\varepsilon$
as $n\delta < b - a$

putting $\varepsilon = \frac{\varepsilon}{b-a}$ we get $\sum_{i=1}^{n} v(P_i'') < \varepsilon$
$\Rightarrow$ measure of $G_f = 0$

6. $f : \mathbb{R} \to \mathbb{R}$
$$f(x) = \begin{cases} e^{-\frac{1}{(x-1)^2}} e^{-\frac{1}{(x+1)^2}} & ; x \in (-1, 1) \\ 0 & ; x \notin (-1, 1) \end{cases}$$

If $a \in \mathbb{R}^n$ $g : \mathbb{R}^n \to \mathbb{R}$
is s.t

$$g(x) = f\left(\frac{x^1 - a^1}{\varepsilon}\right) \cdot f\left(\frac{x^2 - a^2}{\varepsilon}\right) \cdots f\left(\frac{x^n - a^n}{\varepsilon}\right)$$

<u>To prove</u> : $g \in C^\infty(\mathbb{R}^n)$ and $g > 0$ for $(a^1 - \varepsilon, a^1 + \varepsilon) \times \cdots \times (a^n - \varepsilon, a^n + \varepsilon)$

$g = 0$ for all other points

proof : From tutorial-4, Q-2 f is C^∞ is known
and also
$f > 0$ for $x \in (-1,1)$
$f = 0$ for $x \notin (-1,1)$

now, $g: \mathbb{R}^n \longrightarrow \mathbb{R}$

let $f_1(x) = f\left(\frac{x^1 - a^1}{\varepsilon}\right)$

ie $g(x) = f_1(x) f_2(x) \cdots f_n(x)$

now

as $f_1(x) = f\left(\frac{x^1 - a^1}{\varepsilon}\right)$

$f_1(x^1, x^2, \ldots x^n) = f\left(\frac{x^1 - a^1}{\varepsilon}\right)$

$\mathbb{R}^n \longrightarrow \mathbb{R}$

we have

$$f_1'(x) = \left[\frac{f'}{\varepsilon}\left(\frac{x^1 - a^1}{\varepsilon}\right) \; 0 \ldots 0\right]_{1\times n}$$

where all entries are cont as they are 0 or
$\frac{1}{\varepsilon} f'\left(\frac{x^1 - a^1}{\varepsilon}\right) \in C^\infty \Rightarrow$ cont

so f_1' is continious, similarly

$$f_1^{(2)}(x) = \left[\frac{f^{(2)}}{\varepsilon^2}\left(\frac{x^1 - a^1}{\varepsilon}\right) \quad 0 \; . \; . \; 0\right]_{1\times n}$$

is also continious as $f^{(2)}\left(\frac{x^1 - a^1}{\varepsilon}\right)$ is continious

similary

$$f_1^{(n)}(x) = \left[\frac{1}{(\varepsilon)^n} f^{(n)}\left(\frac{x^1 - a^1}{\varepsilon}\right) \quad 0 \; . \; . \quad 0\right]_{1\times n}$$

is cont as $f \in C$

now $\Rightarrow f_1(x) \in C^\infty$

similary $f_i(x) \in C^\infty \; \forall \, i = 1, 2 \ldots n$

now, $g(x) = f_1(x) f_2(x) \cdots f_n(x)$

if $h(x) = h_1(x) h_2(x)$
where $h_1, h_2 \in C^\infty$
then $h \in C^\infty$

for $x_0 \in \mathbb{R}^n$

$\forall \varepsilon > 0, \exists \delta_1 > 0$ and $\exists \delta_2 > 0$

$|h_1(x) - h_1(x_0)| < \varepsilon$
$\forall \; |x - x_0| < \delta_1$

$|h_2(x) - h_2(x_0)| < \varepsilon \; \forall \; |x - x_0| < \delta$

now $|h(x)-h(x_0)|$
$=|h_1(x)h_2(x)-h_1(x_0)h_2(x_0)|$

$=|h_1(x)h_2(x)-h_1(x_0)h_2(x)$
$+h_1(x_0)h_2(x)$
$-h_1(x_0)h_2(x_0)|$

$< |h_2(x)||h_1(x)-h_1(x_0)|$
$+|h_1(x_0)||h_2(x)-h_2(x_0)|$

if $\delta=\min\{\delta_1,\delta_2\}$
then

$|h_2(x)-h_2(x_0)|<\varepsilon$ for $|x-x_0|<\delta$

$\Rightarrow |h_2(x)|<\underbrace{\varepsilon+|h_2(x_0)|}_{=M}$

$\Rightarrow |h_2(x)|<M$

then $|h(x)-h(x_0)|< M(\varepsilon)+|h_1(x_0)|\varepsilon$
for $\varepsilon=\dfrac{\varepsilon}{M+|h_1(x_0)|}$

$\Rightarrow |h(x)-h(x_0)|<\varepsilon$
$\forall |x-x_0|<\delta$

$\Rightarrow$ h is cont on x_0

as $h(x)=h_1(x)h_2(x)$

and h_1,h_2 are cont
$\Rightarrow h$ is cont

by induction

$g=f_1f_2\dots f_n$
as f_1,f_2 is cont
$\Rightarrow f_1f_2$ is cont
similarly $f_1f_2f_3$ is cont
$\dots f_1f_2\dots f_n=g$ is cont

so g is cont, now as $f_i\in C^\infty$ $\forall i$
by product rule

$g'(x)=f_1'(x)f_2(x)\dots f_n(x)$

$+f_1(x)f_2'(x)\dots f_n(x)$
$+\dots$

now $g'(x)$ is also cont as
all f_i and $f_i'\in C^\infty$

now if $h(x)=h_1(x)h_2(x)$
where $h_1,h_2\in C^\infty$
then h is C^∞
as:
$h'(x)=h_1'(x)h_2(x)+h_2'(x)h_1(x)$

$^1C_0$ $^1C_1$ 1
1 2 1
1 3 3 1
⋮

by induction

$h^{(n)}(x)=\sum_{i=0}^{n}({}^nC_i)\,h_1^{(n-i)}(x)\,h_2^{(i)}(x)$

as if $h^{(n-1)}(x) = \sum_{i=0}^{n-1} ({}^{n-1}C_i)(h_1^{(n-1-i)}(x))(h_2^{i}(x))$

$$h^{(n)}(x) = \sum_{i=0}^{n-1} ({}^{n-1}C_i)\left[h_1^{(n-i)}(x)\, h_2^{i}(x) + h_1^{(n-1-i)}(x)\, h_2^{i+1}(x)\right]$$

$$= \sum_{i=0}^{n-1} ({}^{n-1}C_i)\, h_1^{(n-i)} h_2^{(i)} + \sum_{i=0}^{n-1} h_1^{(n-1-i)} h_2^{i+1} ({}^{n-1}C_i)$$

$$= \sum_{i=0}^{n} {}^{n}C_i\, h_1^{(n-i)} h_2^{(i)}$$

now, then $h^{(n)}$ is cont, $\forall n$
$\Rightarrow h \in C^\infty$

now, $h \in C^\infty$ for $h = \underset{\in C^\infty}{h_1}\underset{\in C^\infty}{h_2}$

then $f_1 f_2 \in C^\infty$

$\Rightarrow f_1 f_2 f_3 \in C^\infty$

$\vdots$

$\Rightarrow f_1 f_2 \ldots f_n = g \in C^\infty$

$\therefore g \in C^\infty$

also $g = f_1 f_2 \ldots f_n$

where for $x' \in (-1,1)$

$f_1(x) = f\left(\frac{x' - a'}{\varepsilon}\right)$

for $\frac{x' - a'}{\varepsilon} \in (-1,1)$

$f_1(x) > 0$

$\Rightarrow x' - a' \in (-\varepsilon, \varepsilon)$

$\Rightarrow x' \in (a' - \varepsilon, a' + \varepsilon)$

$\therefore f_1 > 0$ for $x' \in (a' - \varepsilon, a' + \varepsilon)$
else 0

similarly $f_i > 0$ for $x^i \in (a^i - \varepsilon, a^i + \varepsilon)$

so for $g > 0 \Rightarrow f_i > 0\ \forall i$

$\Rightarrow x \in (a^1 - \varepsilon, a^1 + \varepsilon) \times (a^2 - \varepsilon, a^2 + \varepsilon) \times \cdots \times (a^n - \varepsilon, a^n + \varepsilon)$
else 0.

Assignment - 2 :

Dhairya
23B3321

Tutorial-7 :

1. $A \subset \mathbb{R}^n$ closed rectangle
$f, g : A \longrightarrow \mathbb{R}$ is integrable

$\Rightarrow$ let B_1 be set of discontinuities of f on A
B_2 be set of discontinuities of g on A
B_1 and B_2 has measure 0.

now, $f \cdot g : A \longrightarrow \mathbb{R}$
is s.t
if f is cont and g is cont on x
$\Rightarrow f \cdot g$ is cont on x
(using ε-δ definition)
so $\forall x \in A$ s.t f is cont, g is cont $\Rightarrow f \cdot g$ is cont

$\Rightarrow \forall x \in A \setminus B_1 \cup B_2$
$f \cdot g$ is cont

$\Rightarrow B =$ set of points $f \cdot g$ is discont $\subseteq B_1 \cup B_2$

$\Rightarrow B$ also has measure 0

$\Rightarrow f \cdot g$ is integrable

2. To show: $f : [a,b] \longrightarrow \mathbb{R}$ is integrable on $[a,b]$, given it is increasing.

as f is increasing, from lecture 6 :

f is cont at $x \Leftrightarrow O(f,x) = 0$
so f is discont at x then $O(f,x) > 0$

$$B = \{x \mid f \text{ is discont } x\}$$
$$= \{x \mid O(f,x) > 0\}$$
$$= \underbrace{\{x \mid O(f,x) > 1\}}_{B_1} \cup \underbrace{\{x \mid O(f,x) > \tfrac{1}{2}\}}_{B_{\frac{1}{2}}} \cup \dots$$

$$B_{\frac{1}{n}} = \{x \mid O(f,x) > \tfrac{1}{n}\}$$

now $\sum_{x \in B_{\frac{1}{n}}} O(x,f) < f(b) - f(a)$
as f is inc

$\Rightarrow \exists$ finitely many points in $B_{\frac{1}{n}}$

$\Rightarrow$ measure of $B_{\frac{1}{n}} = 0 \ \forall n \in \mathbb{N}$

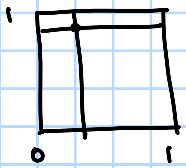
$\Rightarrow B$ has measure 0

$\Rightarrow f$ is integrable

3. $A = [0,1] \times [0,1]$ $f : A \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{1}{q} ; & y \text{ is rational and } x = P/q \text{ in lowest terms} \\ 0 ; & \text{otherwise} \end{cases}$$

(a) To show $\int_A f$ exist, let B = set of points where f is discont



then $O(x,f) > 0$ for those points

$$O(x,f) = \lim_{\delta \to 0} M(x,f,\delta) - m(x,f,\delta)$$

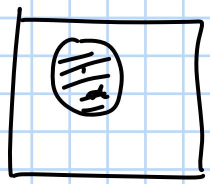
now, if $O(x,f) > \frac{1}{n}$ then

$$\lim_{\delta \to 0} M(x,f,\delta) - m(x,f,\delta) > \frac{1}{n}$$

where

$$m(x,f,\delta) = \min\left\{ f(y) \,\middle|\, |y-x| < \delta \right\}$$

now $|y-x| < \delta \Rightarrow$



$B_\delta(x)$ where

as irrationals are dense

$\exists$ some $(y^1, y^2) \in B_\delta(x)$ s.t

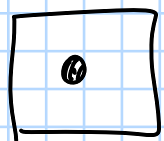
$y^1 \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$

$y^2 \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$

So, $m(x,f,\delta) = 0$

so, $O(x,f) = \lim_{\delta \to 0} M(f,x,\delta) > \frac{1}{n}$

now $\lim_{\delta \to 0} M(f,x,\delta) = \frac{1}{q} > \frac{1}{n}$ for this $x' = P/q$



or else as $\delta \to 0$ this will become contradiction

$\Rightarrow n > q$

or $q = 1, 2, \dots n-1$

then $A_n = \left\{ x \in \mathbb{Q} \cap [0,1] \,\middle|\, x = P/q, (P,q) = 1, q \leq N \right\}$

a $q \leq N$

and $\left\{ \frac{0}{N}, \frac{1}{N}, \dots \frac{N}{N} \right\} \in \mathbb{Q} \cap [0,1]$

finite

similar $\left\{ \frac{0}{N-1}, \dots \frac{N-1}{N-1} \right\}$ finite

and so on

$\Rightarrow A_n$ is finite

so $o(x,f) > \frac{1}{n}$

$\Rightarrow x = (x^1, x^2)$ is s.t

$x^1 \in A_n$

or

$B_{\frac{1}{n}} = \{(x^1,x^2) \mid o(x,f) > \frac{1}{n}\}$

$= \{(x^1,x^2) \mid \underset{\downarrow \text{ finite points}}{x^1 \in A_n}, x^2 \in [0,1]\}$

$B_{\frac{1}{n}} = \{x \mid x^1 \in A_n, x^2 \in [0,1]\}$

is covered by $\underbrace{\{x_k\} \times [0,1]}_{U_i}$ s.t $x_k \in A_n$ as A_n is finite

$\{\{x_k\} \times [0,1] \mid \forall k\}$ covers $B_{\frac{1}{n}}$

is finite

and

$\sum v(U_i) = 0 < \varepsilon \ \forall \varepsilon \Rightarrow$ content $= 0$ for $B_{\frac{1}{n}}$

now $B = B_1 \cup B_{\frac{1}{2}} \cup \ldots\ldots$

has measure 0

$\Rightarrow \int_A f$ exist

(b) $L \int f(x,y)\,dy$

$y \in [0,1]$ for y irrational

$f(x,y) = 0$

and so,

$L \int_{y \in [0,1]} f(x,y) = \sup_P L(f,P)$

$\forall S \in P$ rectangle

$\exists y$ s.t y is irrational (density)

$\Rightarrow L(f,P) = 0 \ \forall P$

$\Rightarrow \sup_P L(f,P) = 0$

$\Rightarrow L \int_{y \in [0,1]} f(x,y)\,dy = 0$ (same for both cases $x \in \mathbb{Q}$ or $x \in \mathbb{R} \setminus \mathbb{Q}$)

now for $U \int_{y \in [0,1]} f(x,y)\,dy$ it is different

as every $S \in P$ we contains (x,y) s.t both are rational

$$U\int_{y\in[0,1]} f(x,y)\,dy = \begin{cases} \frac{1}{q} \ ; & x \text{ is rational and of form } x = p/q,\ (p,q)=1,\ x\in[0,1] \\ 0 \ ; & x \text{ is irrational } x\in[0,1] \end{cases}$$

so, $L\int_{y\in[0,1]} f(x,y)\,dy = 0$

$$U\int_{y\in[0,1]} f(x,y)\,dy = \begin{cases} \frac{1}{q} ; & x\in \mathbb{Q}\cap[0,1],\ x=p/q,\ \gcd(p,q)=1 \\ 0 ; & x\in \mathbb{R}\setminus\mathbb{Q}\cap[0,1] \end{cases}$$

(c) fubini's theorm, f is integrable on $[0,1]\times[0,1]$

$$\mathcal{L}(x) = 0 \quad \forall\, x\in[0,1]$$

$$U(x) = \begin{cases} 1/q ; & x \text{ is rational} \\ 0 ; & x \text{ is irrational} \end{cases}$$

now $\displaystyle\int_{[0,1]\times[0,1]} f = \int_{[0,1]} 0.dx = 0$

and $\displaystyle\int_{[0,1]\times[0,1]} f = \int_{[0,1]} U(x)\,dx$

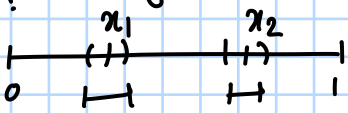
now as there are only finitely many points for $U(x) > \frac{1}{n}$

let $A = \{x \mid U(x) > \frac{1}{n}\} \rightarrow$ finite points

$B = \{x \mid U(x) \le \frac{1}{n}\}$

now for any partion of $U(x)$

let partion be:



$[x_i - \frac{\varepsilon}{2}, x_i + \frac{\varepsilon}{2}]$ is preset $\forall\, x_i\in A$ then

(we pick ε later)

$L(U,P) = 0$

$$U(U,P) = \sum_{S_i} M_{S_i}(U) \le \frac{1}{n}\times 1 + \sum_{x_i\in A} \varepsilon\, U(x_i)$$

$$\Rightarrow U(U,P) \le \frac{1}{n} + \sum \varepsilon\, U(x_i)$$

$$< \frac{1}{n} + \varepsilon |A|$$

cardinality of A

$$\Rightarrow U(U,P) < \frac{1}{n} + \varepsilon|A|$$

putting $\varepsilon = \frac{1}{|A|} \times \frac{1}{n}$

$\Rightarrow U(\vartheta, P) < \frac{2}{n}$

and now $\forall \varepsilon' > 0$

make n s.t

$\varepsilon' > \frac{2}{n}$

$\Rightarrow U(\vartheta, P) < \varepsilon' \ \forall \varepsilon' > 0$

so $U(\vartheta, P) - L(\vartheta, P) < \varepsilon' \ \forall \varepsilon' > 0$

and as

$L(\vartheta, P) = 0 \ \forall P$

$$\Rightarrow \int_{[0,1]} \vartheta = 0$$

$\therefore$ fibins theorm is verified.

4. $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$

cont

$D_2 f$ is cont

$F(y) = \int_a^b f(x,y)\,dy$

<u>To prove</u> : $F'(y) = \int_a^b D_2 f(x,y)$

<u>proof</u> : as $f : [a,b] \times [c,d]$ is cont

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d F(y)\,dy$$

now similarly $\int_a^b \int_c^y D_2 f(x,y)\,dy\,dx$

$$= \int_c^y \int_a^b D_2 f(x,y')\,dx\,dy'$$

$$\int_a^b \int_c^y D_2 f(x,y)\,dy'\,dx = \int_a^b f(x,y) - f(x,c)\,dx$$

$$= F(y) - F(c)$$

so $\int_a^b \int_c^y D_2 f(x,y')\,dy'\,dx = F(y) - F(c)$

now $\int_c^y \int_a^b D_2 f(x,y')\,dx\,dy' = F(y) - F(c)$

($\because$ fubini)

$F'(y) = \int_a^b D_2 f(x,y)\,dx$ ($\because$ FTC)

5. $g_1, g_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \in C^1$

now

$$f(x,y) = \int_0^x g_1(t,0)\,dt + \underbrace{\int_0^y g_2(x,t)\,dt}_{F(x)}$$

$$F(x) = \int_0^y g_2(x,t)\,dt$$

$$= \int_0^y \int_0^x D_1 g_2 \,dx\,dt$$

$$F(x) = \int_0^x \int_0^y D_1 g_2 \,dt\,dx$$

$$\Rightarrow F'(x) = \int_0^y D_1 g_2 \,dt \quad (FTC)$$

$$F'(x) = \int_0^y D_2 g_1 \,dt$$

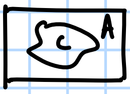
now $\quad D_1 f(x,y) = g_1(x,0) + \int_0^y D_2 g_1$

$$= g_1(x,0) + g_1(x,y) - g_1(x,0)$$

$$D_1 f(x,y) = g_1(x,y)$$

Tutorial-8 :

1. $A \subset \mathbb{R}^n$
↳ closed rectangle
$C \subset A$



f, g bounded functions integrable on C.

$\Rightarrow af + bg$ also integrable over C.
($\because f, g$ cont outside B_1, B_2
$\Rightarrow af + bg$ cont outside $B_1 \cup B_2$)

$$\int_C f = \int_A \chi_C f$$

where $\chi_C(x) = \begin{cases} 0 ; x \in A \setminus C \\ 1 ; x \in C \end{cases}$

$\chi_C : A \longrightarrow \mathbb{R}$

now, $\int_C af + bg$

case I : $a, b \geq 0$

as f is integrable so is af (set of disct. arg)

here $\int_C af = \sup_P L(af, P)$
$= \inf_P U(af, P)$

now $L(af, P) = \sum_{S \in P} m_S(af) v(S)$

$a \geq 0$

now $m_S(af) = a\, m_S(f)$

$\Rightarrow L(af, P) = a\, L(f, P)$

similarly $U(af, P) = a\, U(f, P)$

$\Rightarrow a \sup_P L(f, P) = \sup_P L(af, P)$

$a \int_C f = \int_C af$

similarly $b \int_C g = \int_C bg$

now $\int_C f + g = \int_C f + \int_C g$

proof :

as

$L(f+g, P) = \sum_{S \in P} m_S(f+g) v(S)$

now

$m_S(f+g) \geq m_S(f) + m_S(g)$

$\Rightarrow L(f+g, P) \geq L(f, P) + L(g, P)$

$\Rightarrow \int_C f+g \geq \int_C f + \int_C g$

now, $U(f+g, P) = \sum_{S \in P} M_S(f+g) v(S)$

$$M_S(f+g) \leq M_S(f) + M_S(g)$$

$$\Rightarrow \int_C f+g \leq \int_C f + \int_C g$$

so $\int_C f+g = \int_C f + \int_C g$

for $a, b \geq 0$

now $\int_C af + bg = \int_C af + \int_C bg$

$$= a\int_C f + b\int_C g$$

<u>case II</u> : $a=-1 \quad b=0$

then <u>To prove</u>: $\int_C (-1)f = -\int_C f$

<u>proof</u>:

as f is integrable on C

$\Rightarrow -f$ is integrable on C

now $L(-f,P) = \sum_{S \in P} m_S(-f)\,v(S)$

$m_S(-f) = -M_S(f)$

$\Rightarrow L(-f,P) = -U(f,P)$

$\Rightarrow \int_C -f = -\int_C f$

<u>case III</u>: general case:

$$\int_C \underset{f_1}{\underset{\parallel}{af}} + \underset{g_1}{\underset{\parallel}{bg}} = \int_C \underset{f_1}{\underset{\parallel}{af}} + \int_C \underset{g_1}{\underset{\parallel}{bg}} \quad \text{(from case I)}$$

now $\int_C af = a\int_C f$ if $a \geq 0$

then if $a < 0$

$$\int_C af = \int_C (-a)(-f)$$

$$= -a\int_C (-f)$$

$$= -a(-1)\int_C f$$

$$= a\int_C f$$

so $\int_C af + bg = a\int_C f + b\int_C g$

2. $A \subset \mathbb{R}^n$ closed rectangle $C \subset A$ f, g integrable over C

and $f = g$ except on B (set B has measure 0)

then $h : C \longrightarrow \mathbb{R}$ s.t

let $h(x) = f(x) - g(x)$

now $h(x) = 0$ except for $x \in B$ → set has measure 0

and as f is integrable

f is cont outside B_1 → set of measure 0

similarly g is cont outside B_2 → set of measure 0

so $f - g$ is cont outside $B_1 \cup B_2$

now $h = f - g$ then is cont outside $B_1 \cup B_2$ which is of measure 0

then $\int_C h = \int_C f - g = \int_C f - \int_C g$

now $\int_C h$ is s.t

$h = 0$ except for B → measure 0

now from comparison rule

$|h|$ is integrable and

$|\int_C h| \leqslant \int_C |h| = I$

now $|h|$ is also 0 outside B → measure 0

now $\int_C |h| = I$, $\forall \varepsilon > 0, \exists P$ s.t

$|U(|h|, P) - L(|h|, P)| < \varepsilon$

for every partition if $x \in B$ is in some $S \in P$ → rectangle

and x are countable many

so $\exists y \in S$ s.t $y \notin B$

$\therefore m_S(|h|) = 0$ $\forall S \in P$

$\Rightarrow L(|h|, P) = 0$

$\Rightarrow |I| < \varepsilon$

$\Rightarrow I = 0$

so $\int_C |h| = 0$

$\Rightarrow |\int_C h| \leqslant 0$

$\Rightarrow |\int_C h| = 0$

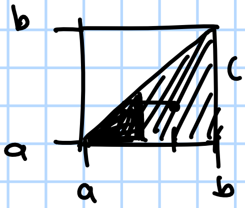
$\Rightarrow \int_C h = 0$

$\Rightarrow \int_C f - \int_C g = 0 \Rightarrow \int_C f = \int_C g$

3. $f : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ is continious

To prove : $\int_a^b \int_a^y f(x,y)\,dx\,dy = \int_a^b \int_x^b f(x,y)\,dy\,dx$

proof : As f is continious
it is also cont on C
$C \subseteq A$ and Bound(C) has measure 0 (trivial)
so $\int_C f$ exist and now



$$\int_C f = \int_A \chi_C f \qquad C = \left\{ (x,y) \middle| \begin{matrix} a \leq x \leq b \\ a \leq y \leq b \end{matrix} ,\ x \leq y \right\}$$

$$\chi_C = \begin{cases} 0 ; & x \in A \setminus C \\ 1 ; & x \in C \end{cases}$$

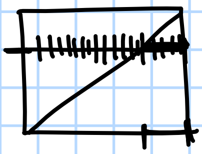
now, $\int_A \chi_C f = \int_C f$

by fubinis theorm, $\int_A \chi_C f = \int_a^b L \int_a^b \chi_C f\,dy\,dx$

$$= \int_a^b L \int_a^b \chi_C f\,dx\,dy$$

now $\int_a^b \chi_C f\,dx$

$$\chi_C f = \begin{cases} 0 & ; a \leq x \leq y \\ f & ; b \geq x > y \end{cases}$$



$$\int_a^b \chi_C f = \int_a^y \chi_C f + \int_y^b \chi_C f$$

$$= 0 + \int_y^b f$$

$$\int_a^b \chi_C f = \int_y^b f$$

then $L \int_a^b \chi_C f\,dx = \int_y^b f\,dx$ as lower sum here will be same as f is cont

so $\int_C f = \int_a^b \int_y^b f\,dx\,dy$

similarly $\int_C f = \int_a^b \int_a^x f\,dy\,dx$

so $\int_a^b \int_a^x f\,dy\,dx = \int_a^b \int_y^b f\,dx\,dy$

4. $g: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation

(a) $g(e_i) = \begin{cases} e_i & ; i \neq j \\ a e_j & ; i = j \end{cases}$

$U = [a_1, b_1] \times \cdots [a_n, b_n]$

$$g: \begin{array}{c} \\ j^{th} \end{array} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & a & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix}$$

j^{th} row j^{th} column

wlog $a > 0$

now $g(U) = [a_1, b_1] \times \cdots \times \underbrace{[a a_j, a b_j]}_{j^{th} \text{ component}} \times \cdots [a_n, b_n]$

now $\int_{g(U)} 1 = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int \cdots \int_{a a_k}^{a b_k} \int \cdots \int_{a_n}^{b_n} 1$ as 1 is cont

by fubinis theorm

$= a[b_1 - a_1][b_2 - a_2] \cdots [b_n - a_n]$

$= a = a\,v(U)$ and $-a$ for $a < 0$

$|\det(g)| = a = |a|$

$v(U) = \prod_{i=1}^{n} (b_i - a_i)$

so $\int_{g(U)} 1 = |\det(g)|\, v(U)$

(b) $g(e_i) = \begin{cases} e_i & ; i \neq j \\ e_j + e_k & ; i = j \end{cases}$

$$g := \begin{array}{c} \\ j^{th} \\ k^{th} \end{array} \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 & & 0 \\ & & & 1 & & \\ & & & 1 & 1 & \\ & 0 & & 0 & 0 & 1 \end{bmatrix}$$

1 here

$\Rightarrow \sum_{i=1}^{n} a_{ij} C_{ij} = \det(g)$ cofactor

$= a_{jj} \det(I_{n-1 \times n-1})$

$+ a_{kj} C_{kj}$ 0 as one row 0

$= 1$

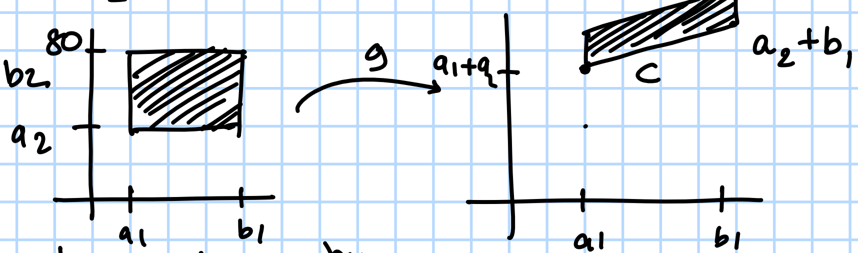
now $U = [a_1, b_1] \times \cdots [a_n, b_n]$

$[a_j, b_j] \times [a_k, b_k]$ is only affected by $g(U)$

for 2×2 case

now $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_{2\times 2}$ $j = 1 \quad k = 2$

$g := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_{2\times 2}$ then $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^1 + x^2 \end{bmatrix}$

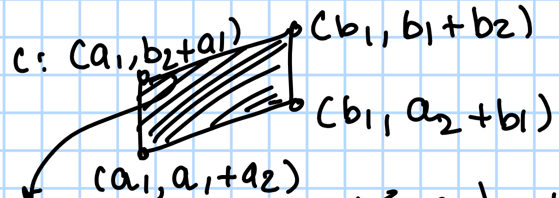


then in general case $\int_{g(U)} 1 = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} \cdots \int_{a_n}^{b_n} \int_C 1$

without a_j, b_j, a_k, b_k

This is by fubini's theorem

now $\int_C 1$



$$\left(\frac{y-b_2-a_1}{x-a_1}\right) = \left(\frac{b_1+\cancel{b_2}-\cancel{b_2}-a_1}{b_1-a_1}\right) = 1$$

$$y = x - \cancel{a_1} + \cancel{a_1} + b_2$$
$$y_2 = x + b_2$$
$$y_1 = x + b_1$$

from 1-D calculus

$$\int_C 1 = \int_{a_1}^{a_2} [y_2 - y_1]\, dx$$
$$= \int_{a_1}^{a_2} (b_2 - b_1)\, dx = (b_2-b_1)(a_2-a_1)$$

then $\int_{g(u)} 1 = v(u) = \underbrace{|\det(g)|}_{=1}\, v(u)$

(c) $g(e_\ell) = \begin{cases} e_k & ; \ell = k \quad k \neq i,j \\ e_j & ; \ell = i \\ e_i & ; \ell = j \end{cases}$

$$g := \begin{array}{c} \\ \\ i\text{th} \\ j\text{th} \end{array} \begin{array}{c} \cdots\cdots\cdots\ g(e_i) \cdots g(e_j) \cdots \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 0 & \cdots & 1 \\ & & & 1 & \cdots & 0 & \cdots \end{bmatrix} \end{array}$$

now $|\det(g)| = \left|\sum_{i=1}^{n} a_{ij} c_{ij}\right|$ ← cofactor

$$= |\underset{a_{ij}}{(1)} \ \underset{(-1)}{c_{ij}}|$$ → property of det (change of columns)

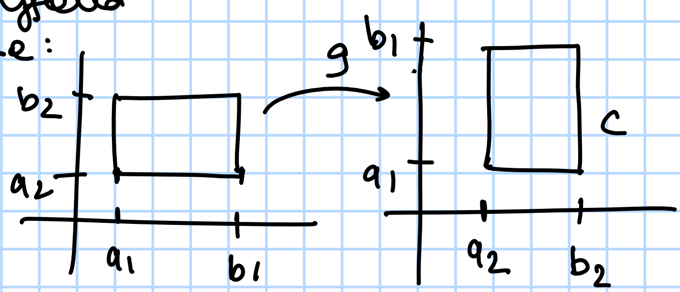
$$= 1$$

now, $u = [a_1, b_1] \times \cdots [a_i, b_i] \times \cdots [a_j, b_j] \times \cdots [a_n, b_n]$

only this two affected

then for a smaller case:

$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then



then $\int_{g(U)} 1 = \underbrace{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n}}_{\text{not invarg } i,\, j} \int_C 1$ $\qquad \int_C 1 = [b_j - a_j] \times [b_i - a_i]$

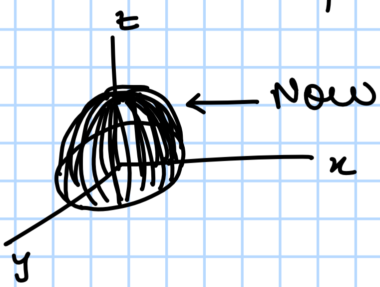
$\downarrow$

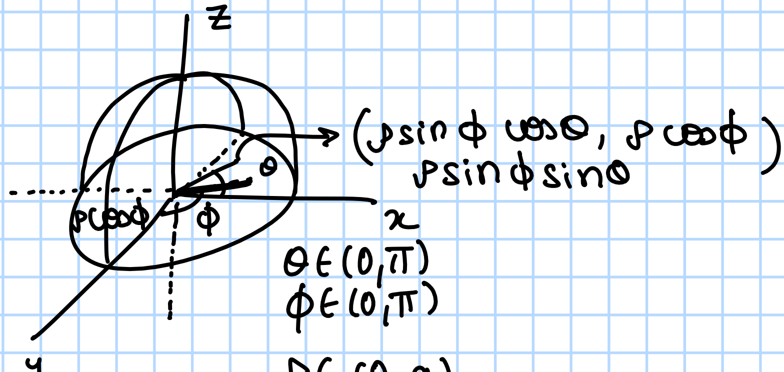
normal case

$$\int_{g(U)} 1 = \vartheta(U) = |\det(g)|\, \vartheta(U)$$

Tutorial-9:

1. $V = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < a^2 \ \& \ z > 0\}$





$\theta \in (0,\pi)$
$\phi \in (0,\pi)$

$\rho \in (0,a)$

now $g(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\cos\phi, \rho\sin\phi\sin\theta)$

now $\tilde{V} = \{(\rho,\phi,\theta) \mid \rho \in (0,a), \theta \in (0,\pi), \phi \in (0,\pi)\}$

open wrt (ρ,ϕ,θ) system

& $V \rightarrow$ given is open in $\mathbb{R}^3$

now $g: \tilde{V} \rightarrow V$ (By construction of g)

$\tilde{V}, V$ is open also as

$$g'(\rho,\phi,\theta) = \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{bmatrix}_{3\times3}$$

so $g'(\rho,\phi,\theta)$ all a_{ij} are continious

$\Rightarrow g$ is continious differentiable

and also

$$\det g'(\rho,\phi,\theta) = -\rho^2\sin^3\phi\cos^2\theta - \rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\sin^3\phi\sin^2\theta - \rho^2\cos^2\phi\sin\phi\cos^2\theta$$

$$= -\rho^2\sin^3\phi - \rho^2\cos^2\phi\sin\phi$$
$$= -\rho^2\sin\phi$$

now as $\rho^2 > 0$
$\sin\phi > 0$
as $\phi \in (0,\pi)$

$\Rightarrow \det g' \neq 0 \ \forall (\rho,\phi,\theta) \in \tilde{V}$

so By inverse function theorm,
g is 1-1 and onto
and $(g^{-1})'$ is cont

or $g \in C^1$
& $g^{-1} \in C^1$

$\Rightarrow g$ is diffeomorphism

from $g: \tilde{V} \rightarrow V$

now change of variables tells us that if g is a diff from

$g : A \longrightarrow B$

& $f : B \longrightarrow \mathbb{R}$ is cont then

$\int_B f$ int iff $\int_A (f \circ g)|\det g|$ int

and if so then $\int_B f = \int_A (f \circ g)|\det g|$

now in our case

$f = z : V \longrightarrow \mathbb{R}$ is continious (trivial)

also

$f \circ g = (\rho \sin\phi \sin\theta)$

$\int_A \rho\sin\phi\sin\theta(\rho^2 \sin\phi)$

$$= \int_0^a \int_0^\pi \int_0^\pi \rho^3 \sin^2\phi \sin\theta \, d\theta \, d\phi \, d\rho$$

($\because$ fubini's theorm)

so $\int_A f \circ g |\det g|$ is integrable

$\Rightarrow \int_B f$ is integrable and

$\int_B f = \int_A (f \circ g)|\det g|$

now $\int_B z = \int_0^a \int_0^\pi \int_0^\pi \rho^3 \sin^2\phi \sin\theta \, d\theta \, d\phi \, d\rho$

$$= \int_0^a \int_0^\pi \rho^3 \sin^2\phi \, (2) \, d\phi \, d\rho$$

$$= \int_0^a \int_0^\pi \rho^3 (1 - \cos 2\phi) \, d\phi \, d\rho$$

$$= \int_0^a \pi \rho^3 \, d\rho$$

$$= \frac{\pi a^4}{4}$$

$1 + \cos\theta = 2\cos^2 \frac{\theta}{2}$

$1 - \cos\theta = 2\sin^2 \frac{\theta}{2}$

$1 - \cos 2\phi = 2\sin^2\phi$

2. B portion of first quad of $\mathbb{R}^2$

To find $\int_B x^2 y^3$ now $x = \frac{u}{v}$ $y = uv$

then

$g(u,v) = \left(\frac{u}{v}, uv\right)$

now $g'(u,v) = \left| \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix} \right| = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$

now well $x = \frac{u}{v}$ as in First Quad $x > 0$

now let $g : A \longrightarrow B$ Be s.t

$A = \{ (u,v) \mid 1 < u < \sqrt{2},\ 1 < v < 2 \}$

then A is open, B is open and By construction we made A

now, $g(u,v) = \left(\frac{u}{v}, uv \right)$

as $g'(u,v) = \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix}$ a_{ij} is cont

$\Rightarrow$ g is cont diff

now as $\det(g') = \frac{2u}{v} > 0 \ \forall (u,v) \in A$

By inverse function theorm $g : A \longrightarrow B$ is s.t

g is one-one, onto

& $g \in C^1$, $g^{-1} \in C^1$

$\Rightarrow$ g is diff.

now, $f : B \longrightarrow \mathbb{R}$ is s.t

$f = x^2 y^3$

$x^2 y^3 : B \longrightarrow \mathbb{R}$ is cont (trivial)

now, $\int_A (f \circ g) |\det g|$

$= \int_A \left(\frac{u}{\cancel{v}} \right)^2 (u\cancel{v})^3 \left(2\frac{u}{\cancel{v}} \right)$

$= \int_A 2u^6$ by fubinis theorm

$= \int_1^{\sqrt{2}} \int_1^2 2u^6 \, dv \, du$

then By change of variables

$\int_B f = \int_1^{\sqrt{2}} \int_1^2 2u^6 \, dv \, du$

$= \int_1^{\sqrt{2}} 2u^6 \, du$

$= \frac{2}{7} [(\sqrt{2})^7 - 1] = \frac{2}{7} [8\sqrt{2} - 1]$

3. S = Tetrahedron in $\mathbb{R}^3$ with $A(0,0,0)$
$B(1,2,3)$
$C(0,1,2)$
$D(-1,1,1)$

now if we make g as a liner trasformation s.t

$A'(0,0,0) \xrightarrow{g} A(0,0,0)$
$B'(1,0,0) \xrightarrow{g} B$
$C'(0,1,0) \xrightarrow{g} C$
$D'(0,0,1) \xrightarrow{g} D$

now then G ← matrix rep of g

$$G\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

then $G := \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

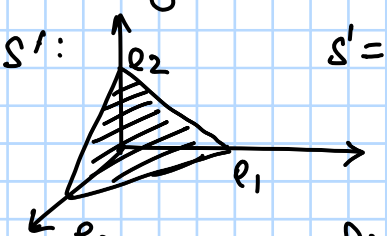
now $|\det(G)| = 2 > 0$ so G is invertible

& $\therefore$ $g : S' \longrightarrow S$ is a liner trasformation with $\det G \neq 0$
$\Rightarrow$ g is 1-1, onto
& it is trivial $g \in C'$, $g^{-1} \in C'$

$\Rightarrow$ g is a diff.

now $g : S' \longrightarrow S$

S':



$S' = \left\{ (x,y,z) \middle| \begin{array}{l} 1-x-y-z < 1 \\ x>0, y>0, z>0 \end{array} \right\}$

S' is open & S is open

By costrution
$g : S' \longrightarrow S$ is one-one & onto

now $\int_{S'} (f \circ g) |\det g|$
$= \int_{S'} f(g(x,y,z))(2)$

$= \int_{S'} (2x)(2) = 4 \int_{S'} x$

now as S' is defined as above
by fubinis theorm

$$4\int_{S'} x = 4\int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx$$

$$= 4\int_0^1\int_0^{1-x} x(1-x-y)\,dy\,dx$$

$$= 4\int_0^1 x(1-x) - x^2(1-x) - \frac{x}{2}(1-x)^2\,dx$$

$$= 4\int_0^1 x - x^2 - x^2 + x^3 - \frac{x}{2}(1+x^2-2x)\,dx$$

$$= 4\int_0^1 \frac{x}{2} - x^2 - \cancel{x^2} + \frac{x^3}{2} - \cancel{\frac{x}{2}} - \cancel{\frac{x^3}{2}} + \cancel{x^2}\,dx$$

$$= 4\left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right]$$

$$= 1 - \frac{4}{3} + \frac{1}{2}$$

$$= \frac{3}{2} - \frac{4}{3} = \frac{9-8}{6} = \frac{1}{6}$$

then By chge of variables

$$\int_S f = \int_{S'} (f\circ g)|\det g| = \frac{1}{6}$$

4. (a) $f(x,y) = 3x^1y^2 + 5x^2x^3$

$$f(x, y_1+y_2) = 3x^1[y_2^2 + y_1^2] + 5x^2x^3$$

$$= 3x^1y_2^2 + 5x^2x^3 + 3x^1y_1^2$$

$$\neq f(x,y_1) + f(x,y_2)$$

$\therefore$ not multilinear

$\therefore$ not a tensor

(b) $g(x,y) = x^1y^2 + x^2y^4 + 1$

$$g(x_1+x_2, y) = x_1^1y^2 + x_2^1y^2 + x_1^2y^4 + x_2^2y^4 + 1$$

$$\neq g(x_1,y) + g(x_2,y)$$

$\therefore$ not a tensor

(c) $h(x,y,z) = 3x^1x^2z^3 - x^3y^1z^4$

$$h(x, y_1+y_2, z) = 3x^1x^2z^3 - x^3y_1^1z^4 - x^3y_2^1z^4$$

$$\neq h(x,y_1,z) + h(x,y_2,z)$$

$\therefore$ not a tensor

(d) $T(x,y,z) = x^1y^2z^4 + 2x^1z^3$

$T(x, y_1+y_2, z) = x^1y_1^2z^4 + x^1y_2^2z^4 + 2x^1z^3$

Not a tensor $\neq T(x, y_1, z) + T(x, y_2, z)$

5. (a) $f(x,y) = x^1y^2 - x^2y^1 + x^1y^1$

now $f \in T^2(\mathbb{R}^4)$ as

$f = \varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 + \varphi_1 \otimes \varphi_1 \in$ Span of Basis of tensors $T^2(\mathbb{R}^4)$

now $f(y,x) = y^1x^2 - y^2x^1 + x^1y^1$

$-f(x,y) = -y^2x^1 + x^2y^1 - x^1y^1$
$\neq f(y,x)$

$\therefore$ not alternating tensor

(b) $g(x,y) = x^1y^3 - x^3y^1$

now $g = \varphi_1 \otimes \varphi_3 - \varphi_3 \otimes \varphi_1$
$\in T^2(\mathbb{R}^4)$

now

$g(y,x) = x^3y^1 - y^3x^1$
$= -g(x,y)$

$\therefore$ is alternating tensor

(c) $h(x,y) = (x^1)^3(y^2)^3 - (x^2)^3(y^1)^3$

now

$h(x_1+x_2, y) = (x_1^1 + x_2^1)^3(y^2)^3 - (x_1^2 + x_2^2)^3(y^1)^3$
$\neq h(x_1,y) + h(x_2,y)$

$\therefore h$ is not a tensor

6. $f: V \longrightarrow W$ is a linear transformation

To prove : If $T \in \Lambda^k(W)$ then $f^*T \in \Lambda^k(V)$

proof : By defination for $T \in T^k(W)$

$(f^*T)(v_1, v_2 \ldots v_k) = T(f(v_1), f(v_2) \ldots f(v_k))$

$\in T^k(V)$ now, as $T \in \Lambda^k(V)$

$(f^*T)(v_1, \ldots v_k) = T(f(v_1), \ldots f(v_k))$

is s.t

$(f^*T)(v_i, v_j) = T(f(v_i), f(v_j))$ where $T \in \Lambda^k(W)$

now

$(f^*T)(v_j, v_i) = T(f(v_j), f(v_i))$

$$= -T(f(v_i), f(v_j))$$

$$(\because T \in \Lambda^k(W))$$

$$= -(f^*T)(v_i, v_j)$$

so $\forall\, i, j \quad i \neq j$

$$(f^*T)(v_i, v_j) = -(f^*T)(v_j, v_i)$$

$$\Rightarrow f^*T \in \Lambda^k(V)$$

Assignment - 3

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23B3321

Tutorial – 10 :

1. **To prove** : $T \in T^k(V)$ for some vectorspace then $Alt(T) = 0$

proof : Now

$$Alt(T)(v_1, v_2 \dots v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma(1)}, v_{\sigma(2)}, \dots v_{\sigma(k)})$$
$v_i \in V$

given T is symmetric, i.e for i, j $i \neq j$

$$T(v_1, \dots v_i, \dots v_j, \dots v_k)$$
$$= T(v_1, \dots v_j, \dots v_i, \dots v_k)$$

we have

$$Alt(T)(v_1, \dots v_j, \dots v_i, \dots v_k)$$
$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \underbrace{T(v_{\sigma(1)}, \dots v_{\sigma(j)}, \dots v_{\sigma(i)}, \dots v_{\sigma(k)})}_{= T(v_{\sigma(1)}, \dots v_{\sigma(i)}, \dots v_{\sigma(j)} \dots v_{\sigma(k)})}$$

By property of symmetry

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma(1)}, \dots v_{\sigma(i)}, \dots v_{\sigma(j)}, \dots v_{\sigma(k)})$$
$$= Alt(T)(v_1, \dots v_i, \dots v_j, \dots v_k) \quad \text{—— } ①$$

now by property of $Alt(T) \in \Lambda^k(V)$ (alternating tensors)

$$Alt(T)(v_1, \dots v_i, \dots v_j, \dots v_k) = -Alt(T)(v_1, \dots v_j, \dots v_i, \dots v_k) \quad ②$$
$i \neq j$

$\Rightarrow Alt(T)(v_1, \dots v_i, \dots v_j, \dots v_k) = 0 \quad (\because ①, ②)$

$\Rightarrow Alt(T) = 0$

now if $T \in \Lambda^3(\mathbb{R}^2)$

say $T(x, y, z) = x^1 y^2 z^1 + x^2 y^1 z^1$

then $T = \varphi_1 \otimes \varphi_2 \otimes \varphi_1 + \varphi_2 \otimes \varphi_1 \otimes \varphi_1$

$\in \text{span} \{ \varphi_{i_1} \otimes \varphi_{i_2} \otimes \varphi_{i_3} \mid 1 \leq i_1, i_2, i_3 \leq 2 \}$

$= \Lambda^3(\mathbb{R}^2)$

now $T = [\varphi_1 \otimes \varphi_2 + \varphi_2 \otimes \varphi_1] \otimes \varphi_1$

where $\varphi_1 \otimes \varphi_2 + \varphi_2 \otimes \varphi_1 \in \Lambda^2(\mathbb{R}^2)$

$\varphi_1 \in \Lambda^1(\mathbb{R}^2)$

now let $S = \varphi_1 \otimes \varphi_2 + \varphi_2 \otimes \varphi_1$

then

$$S(x, y) = x^1 y^2 + x^2 y^1$$
$$S(y, x) = x^2 y^1 + x^1 y^2 = S(x, y)$$

so S is symmetric

so $Alt(S) = 0$

from prev calculation

now then $T = S \otimes \varphi_1$

where $Alt(S) = 0$
so By theorm

$Alt(T) = 0$

But T is not symmetric as

$T(x, y, z) = x^1 y^2 z^1 + x^2 y^1 z^1$
$T(x, z, y) = x^1 z^2 y^1 + x^2 z^1 y^1 \neq T(x, y, z)$

so $Alt(T) = 0 \not\Rightarrow T$ is symmetric

2.(a) $F = 2\varphi_2 \otimes \varphi_2 \otimes \varphi_1 + \varphi_1 \otimes \varphi_5 \otimes \varphi_4$

(b) $G = \varphi_1 \otimes \varphi_3 + \varphi_3 \otimes \varphi_1$

(c) $Alt(F) = Alt(2\varphi_2 \otimes \varphi_2 \otimes \varphi_1) + Alt(\varphi_1 \otimes \varphi_5 \otimes \varphi_4)$
$= 0 + \varphi_1 \wedge \varphi_5 \wedge \varphi_4 \left[\frac{1}{3!}\right]$

$Alt(F) = -\frac{1}{6} \varphi_1 \wedge \varphi_4 \wedge \varphi_5$

(d) $Alt(G) = \varphi_1 \wedge \varphi_3 \left(\frac{1}{2!}\right) + \varphi_3 \wedge \varphi_1 \left(\frac{1}{2!}\right)$
$= 0$

3.(a) $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} = \frac{(\overbrace{1+1+1 \ldots 1}^{k \text{ times}})}{(1)!(1)! \ldots (1)!} Alt(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \varphi_{i_k})$

$= k! \, Alt(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k})$

$= \cancel{k!} \left[\frac{1}{\cancel{k!}} \sum_{\sigma \in S_k} \varphi_{i_{\sigma(1)}} \otimes \varphi_{i_{\sigma(2)}} \cdots \otimes \varphi_{i_{\sigma(k)}} \times sgn(\sigma) \right]$

now $\varphi_{i_1} \wedge \varphi_{i_2} \ldots \wedge \varphi_{i_k} (e_{i_1}, e_{i_2}, \ldots e_{i_k}) = \sum_{\sigma \in S_k} \varphi_{i_{\sigma(1)}} \otimes \cdots \otimes \varphi_{i_{\sigma(k)}} (e_{i_1}, \ldots e_{i_k})$

$= 0 + 1 \rightarrow \sigma = (1, 2, \ldots k) \times (1) \rightarrow sgn(\sigma)$

$\rightarrow$ if $\sigma \neq (1, 2, \ldots k)$

$= 1$

if $\frac{(k+l)!}{k!\, l!}$ did not apper in $\wedge$, then right side will be $\frac{1}{k!}$

(b) $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} (v_1, \ldots v_k)$

$= k! \, Alt(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \varphi_{i_k})(v_1, v_2, \ldots v_k)$

$= \sum_{\sigma \in S_k} \varphi_{i_{\sigma(1)}} \otimes \varphi_{i_{\sigma(2)}} \otimes \cdots \varphi_{i_{\sigma(k)}} (v_1, v_2, \ldots v_k)(sgn(\sigma))$

$= \sum_{\sigma \in S_k} sgn(\sigma) \, v_1^{i_{\sigma(1)}} v_2^{i_{\sigma(2)}} v_3^{i_{\sigma(3)}} \ldots v_k^{i_{\sigma(k)}}$

now $\begin{bmatrix} v_1^1 & v_1^2 & \cdots & v_1^n \\ v_2^1 & v_2^2 & \cdots & v_2^n \\ \vdots & & & \\ v_k^1 & v_k^2 & \cdots & v_k^n \end{bmatrix}$ $k \times k$ minor of it wrt $i_1, i_2 \ldots i_k$ is

$= \sum_{\sigma \in S_k} sgn(\sigma) \, v_1^{i_{\sigma(1)}} v_2^{i_{\sigma(2)}} \ldots v_k^{i_{\sigma(k)}}$

$= \varphi_{i_1} \wedge \varphi_{i_2} \ldots \wedge \varphi_{i_k} (v_1, v_2, \ldots v_k)$

4. $\omega \wedge \varphi \wedge \eta = \omega \wedge (\varphi \wedge \eta)$ φ has degree p

$$= \omega \wedge (-1)^{pt} (\eta \wedge \varphi)$$
$$= (-1)^{pt}\, \omega \wedge \eta \wedge \varphi$$
$$= (-1)^{pt} (-1)^{rt}\, \eta \wedge \omega \wedge \varphi$$
$$= (-1)^{pt} (-1)^{rt} (-1)^{pr}\, \eta \wedge \varphi \wedge \omega$$
$$= (-1)^{pt+rt+pr}\, \eta \wedge \varphi \wedge \omega$$
$$= (-1)^{p\underbrace{[t+r]}_{\text{even}}+rt}\, \eta \wedge \varphi \wedge \omega$$
$$= (-1)^{rt}\, \eta \wedge \varphi \wedge \omega$$

$r = 2k+1$
$t = 2k'+1$
$rt = 2k'+1$
so $(-1)^{rt} = -1$
$\therefore \omega \wedge \varphi \wedge \eta = -\eta \wedge \varphi \wedge \omega$

5. $A \subseteq \mathbb{R}^n$ open

(a) <u>To prove</u> : $\Omega^k(A)$ is a vector space

<u>proof</u> : $\Omega^k(A)$ = collection of all differential k forms of class c

for $\forall P \in A$, if $\omega \in \Omega^k$ then

$\omega(P) \in \Lambda^k(\mathbb{R}^n_P)$

and $\omega(P) = \sum_{1 \leq i_1 < i_2 .. i_k \leq n} \omega_{i_1, i_2 \dots i_k}(P) \left(\varphi_{i_1}(P) \wedge \varphi_{i_2}(P) \dots \varphi_{i_k}(P)\right)$

$\in C^\infty \ \forall P \in A$

now if $\omega \in \Omega^k$, $\eta \in \Omega^k$ then

$$(\omega + \eta)(P) = \omega(P) + \eta(P)$$
$$= \sum \omega_{i_1, i_2 .. i_k}(P) \left(\varphi_{i_1}(P) \wedge \dots \varphi_{i_k}(P)\right)$$
$$+ \sum \eta_{i_1, i_2 .. i_k}(P) \left(\varphi_{i_1}(P) \wedge .. \varphi_{i_k}(P)\right)$$
$$= \sum \left(\omega_{i_1, i_2 \dots i_k} + \eta_{i_1, i_2, \dots i_k}\right)(P) \left(\varphi_{i_1}(P) \wedge \dots \varphi_{i_k}(P)\right)$$

now as $\omega_{i_1, i_2 \dots i_k} : A \longrightarrow \mathbb{R}$ & $\in C^\infty$
$\eta_{i_1, i_2 \dots i_k} : A \longrightarrow \mathbb{R}$ & $\in C^\infty$

$\Rightarrow \omega_{i_1, i_2 .. i_k} + \eta_{i_1, i_2, \dots i_k} : A \longrightarrow \mathbb{R}$
& $\in C^\infty$
(trivial to see)

also for a scalar $\alpha \in \mathbb{R}$
$\alpha\omega \in \Omega^k(A)$ as
$\alpha \omega_{i_1, i_2 \dots i_k} : A \longrightarrow \mathbb{R} \in C^\infty$
as $\omega_{i_1 \dots i_k} \in C^\infty$

now it is similar to show:

for $U, V, W \in \Omega^k(A)$
$\alpha, \beta \in \mathbb{R}$

① $0 \cdot v + v = v$
② $v + (-1)v = 0$
③ $u + (v + w) = (u + v) + w$
④ $v + w = w + v$
⑤ $1 \cdot v = v$
⑥ $\alpha(\beta v) = (\alpha\beta)v$
⑦ $(\alpha + \beta)(v) = \alpha v + \beta v$
⑧ $\alpha(v + w) = \alpha v + \alpha w$

(b) To prove : $B = \left\{ F \;\middle|\; \begin{array}{l} \forall P \in A,\ F(P) \in \mathbb{R}^n_P \\ F(P) = F^1(P)(e^1)_P + \cdots F^n(P)(e^n)_P \\ \forall i,\ F^i(P) \in C^\infty \end{array} \right\}$

is a vector space

proof :

for $F, G \in B$

$$(F + G)(P) = \sum F^i(P)(e^i)_P + \sum G^i(P)(e^i)_P$$

$$= \sum (F^i + G^i)(P)(e^i)_P$$

$\in B$

as $(F^i + G^i)(P) \in C^\infty$

also $\alpha \in \mathbb{R}$

$$\alpha F(P) = \sum \underbrace{\alpha F^i(P)}_{\in C^\infty} (e^i)_P$$

as $F^i(P) \in C^\infty$

$\Rightarrow \alpha F \; \alpha \; B$

trivially we can show:

for $U, V, W \in B$
$\alpha, \beta \in \mathbb{R}$

① $0 \cdot v + v = v$
② $v + (-1)v = 0$
③ $u + (v + w) = (u + v) + w$
④ $v + w = w + v$
⑤ $1 \cdot v = v$
⑥ $\alpha(\beta v) = (\alpha\beta)v$
⑦ $(\alpha + \beta)(v) = \alpha v + \beta v$
⑧ $\alpha(v + w) = \alpha v + \alpha w$

Tutorial-11:

1. (a) $f: \mathbb{R}^n \to \mathbb{R}^m$

$g: \mathbb{R}^m \to \mathbb{R}^p$

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

$$g \circ f : \mathbb{R}^n \to \mathbb{R}^p$$

To prove : $(g\circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$

proof : now if $f: \mathbb{R}^n \to \mathbb{R}^m$

$f_*: \mathbb{R}^n_\gamma \to \mathbb{R}^m_{f(\gamma)}$

s.t $f_*(v_\gamma) = (Df(\gamma)(v))_{f(\gamma)}$

$Df(\gamma) : \mathbb{R}^n \to \mathbb{R}^m$

now

$(g\circ f)_* : \mathbb{R}^n_\gamma \to \mathbb{R}^p_{(g\circ f)(\gamma)}$

s.t $(g\circ f)_*(v_\gamma) = (D(g\circ f)(\gamma)v)_{(g\circ f)(\gamma)}$

$$= \Big(\underset{p\times m}{D(g)(f(\gamma))} \circ \underbrace{\underset{m\times n}{D(f)(\gamma)} \overset{\in \mathbb{R}^n}{v}}_{\in \mathbb{R}^m} \Big)_{(g\circ f)(\gamma)}$$

$$= (D(g)(f(\gamma))\,(D(f)(\gamma)v))_{g(f(\gamma))}$$

$$= g_* \circ (D(f)(\gamma)v)_{f(\gamma)}$$

now $(D(f)(\gamma)v)_{f(\gamma)} = f_*(v_\gamma)$

$$= g_* \circ f_*(v_\gamma)$$

$$(g\circ f)_* = g_* \circ f_*$$

now $f: \mathbb{R}^n \to \mathbb{R}^m$

$f^*: \Lambda^k(\mathbb{R}^m_{f(p)}) \to \Lambda^k(\mathbb{R}^n_p)$

then $(f^*\omega)(p) \in \Lambda^k(\mathbb{R}^n_p)$

s.t

$$(f^*\omega)(p) = f^*(\underbrace{\omega(f(p))}_{\in \Lambda^k(\mathbb{R}^m_{f(p)})})$$

$$(f^*\omega)(p)(v_1,\dots v_k) = \underset{\in \Lambda^k(\mathbb{R}^m_{f(p)})}{\omega(f(p))}\,(\overset{\in \mathbb{R}^m_{f(p)}}{f_*(v_1), f_*(v_2) \dots f_*(v_k)})$$

$v_1, v_2 \dots v_k \in \mathbb{R}^n_p$

$$(g\circ f)^* : \Lambda^k(\mathbb{R}^p_{(g\circ f)(\gamma)}) \to \Lambda^k(\mathbb{R}^n_\gamma)$$

$$\omega(g\circ f)(\gamma) \in \Lambda^k(\mathbb{R}^p_{(g\circ f)(\gamma)})$$

$$((g\circ f)^*\omega)(\gamma)(\underset{\in \mathbb{R}^n_\gamma}{v_1, \dots v_k}) = \underbrace{\omega((g\circ f)(\gamma))}_{\in \Lambda^k(\mathbb{R}^p_{(g\circ f)(\gamma)})}\,(\underset{\in \mathbb{R}^p_{(g\circ f)(\gamma)}}{(g\circ f)_*(v_1), \dots (g\circ f)_*(v_k)})$$

$$= (g^*\omega)(f(\gamma))(f_*v_1, f_*v_2 \dots f_*v_k)$$

$= (f^* \circ g^*)(\omega)(\gamma)(\vartheta_1, \ldots \vartheta_k)$

$\Rightarrow (g \circ f)^* = f^* \circ g^*$ also if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
$d(f) = \sum_{\alpha=1}^{n} D_\alpha f \, dx^\alpha$

2. (a) $x^2 y \, dy - x y^2 dx \in \Lambda^1(\mathbb{R}^2)$

if $\omega = \sum_I \omega_I \, dx^I$
then
$$d\omega = \sum_I \sum_i \partial_i(\omega_I) dx^i \wedge dx^I$$

now let $\omega = x^2 y \, dy - x y^2 dx$

then
$$d\omega = \sum_{i=1}^{2} \partial_i (x^2 y) dx^i \wedge dy - \sum_{i=1}^{2} \partial_i (x y^2) dx^i \wedge dx$$

$$= 2xy \, dx \wedge dy - 2xy \, dy \wedge dx$$
$$d\omega = 4xy \, dx \wedge dy$$

(b) $\omega = \cos(xy^2) dx \wedge dz \in \Lambda^2(\mathbb{R}^3)$

$$d\omega = \sum_I \sum_{i=1}^{3} \partial_i (\omega_I) dx^I$$

$$= \sum_{i=1}^{3} \partial_i (\cos(xy^2)) dx^i \wedge dx \wedge dz$$

$$= \frac{d}{dy} (\cos(xy^2)) dy \wedge dx \wedge dz$$

$$= -\frac{d}{dy} (\cos(xy^2)) dx \wedge dy \wedge dz$$

$$= \sin(xy^2)(2xy) \, dx \wedge dy \wedge dz$$

$$= 2xy \sin(xy^2) \, dx \wedge dy \wedge dz$$

(c) $\omega = f(x, z) dx$

then for $\omega = \sum_I \omega_I dx^I$ if $\in \Lambda^1(\mathbb{R}^2)$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$
then
$D f := [D_1 f \;\; D_2 f]_{1 \times 2}$

now $d\omega = \sum_{i=1}^{2} D_i (f(x, z)) dx^i \wedge dx$

$$d\omega = D_2 (f(x, y)) dz \wedge dx$$

(d) $\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$
then by $d(\omega + \eta) = d(\omega) + d(\eta)$

$$d(\omega) = d(x \, dy \wedge dz) + d(y \, dz \wedge dx) + d(z \, dx \wedge dy)$$
$$= dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy$$

$$= dx\wedge dy\wedge dz + dx\wedge dy\wedge dz + dx\wedge dy\wedge dz$$

$$d(w) = 3\,dx\wedge dy\wedge dz$$

3. $\omega = xy\,dx + 3dy - yz\,dz$
$\eta = x\,dx - yz^2dy + 2x\,dz$

now $d\omega = x\,dy\wedge dx + 0 - z\,dy\wedge dz$

$d(d\omega) = 0 + 0 - 0$
$= 0$

also, $\omega\wedge\eta = (xy\,dx + 3dy - yz\,dz)\wedge(x\,dx - yz^2dy + 2x\,dz)$

$= xy\,dx\wedge(x\,dx - yz^2dy + 2x\,dz)$
$+ 3dy\wedge(x\,dx - yz^2dy + 2x\,dz)$
$+ (-yz)dz\wedge(x\,dx - yz^2dy + 2x\,dz)$

$= -xy^2z^2\,dx\wedge dy + 2x^2y\,dx\wedge dz$
$+ 3x\,dy\wedge dx + 6x\,dy\wedge dz$
$- xyz\,dz\wedge dx + y^2z^3\,dz\wedge dy$

$d(\omega\wedge\eta) = -2xy^2z\,dz\wedge dx\wedge dy + 2x^2\,dy\wedge dx\wedge dz$
$+ 0 + 6\,dx\wedge dy\wedge dz$
$- xz\,dy\wedge dz\wedge dx + 0$

$= (-2xy^2z - 2x^2 + 6 - xz)\,dx\wedge dy\wedge dz$

$d(\omega) = x\,dy\wedge dx + 0 - z\,dy\wedge dz$
$d\omega = x\,dy\wedge dx - z\,dy\wedge dz$
$\eta = x\,dx - yz^2dy + 2x\,dz$
$\omega = xy\,dx + 3dy - yz\,dz$
$d\eta = 0 - 2yz\,dz\wedge dy + 2\,dx\wedge dz$

now $(d\omega)\wedge\eta - \omega\wedge d\eta$

$= [x\,dy\wedge dx - z\,dy\wedge dz]\wedge[x\,dx - yz^2dy + 2x\,dz]$
$- [xy\,dx + 3dy - yz\,dz]\wedge[-2yz\,dz\wedge dy + 2dx\wedge dz]$

$= 2x^2\,dy\wedge dx\wedge dz - xz\,dy\wedge dz\wedge dx$
$- [-2xy^2z\,dx\wedge dz\wedge dy + 6\,dy\wedge dx\wedge dz]$

$= -2x^2\,dx\wedge dy\wedge dz - xz\,dx\wedge dy\wedge dz$
$- [2xy^2z\,dx\wedge dy\wedge dz - 6\,dx\wedge dy\wedge dz]$

$= d(\omega\wedge\eta)$

4. $(n-1)$-form ξ $\quad d\xi = dx^1 \wedge \cdots \wedge dx^n$

now as $d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$

$$= 3\,dx\wedge dy\wedge dz$$

for $\xi = \dfrac{x^1\,dx^2\wedge dx^3 - x^2\,dx^1\wedge dx^3 + x^3\,dx^1\wedge dx^2}{3}$

then $d\xi = dx^1\wedge dx^2\wedge dx^3$

now for K-1 even

$$\xi = \frac{x^1\,dx^2\wedge dx^3\wedge dx^4 - x^2\,dx^1\wedge dx^3\wedge dx^4 + x^3\,dx^1\wedge dx^2\wedge dx^4 - x^4\,dx^1\wedge dx^2\wedge dx^3}{4}$$

so for general case:

$$\xi = \frac{x^1 dx^2\wedge dx^3 \cdots \wedge dx^n + \cdots + (-1)^{i+1} x^i dx^1 \cdots \wedge dx^{i-1}\wedge dx^i \cdots dx^n + (-1)^{n+1} x^n dx^1 \wedge \cdots \wedge dx^{n-1}}{n}$$

then $d\xi = \dfrac{\sum^n dx^1\wedge dx^2 \cdots \wedge dx^n}{n}$

$= dx^1 \wedge \cdots \wedge dx^n$

we are choosing sign s.t when we take $d\xi$ all signs become positive if we arrage

as $-dx^2\wedge dx^1\wedge dx^3 \cdots \wedge dx^n$
$= dx^1\wedge dx^2 \cdots \wedge dx^n$

5. $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$\underbrace{(\text{grad } f)(P)}_{\text{vectorfield}} \in \mathbb{R}^n_P$

$(\text{grad } f)(P) = D_1 f(P)\,(e_1)_P + \cdots D_n f(P)(e_n)_P$

F is vector field on $\mathbb{R}^3$

$\omega^1_F = F^1 dx + F^2 dy + F^3 dz$

$\omega^2_F = F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy$

(a) <u>To prove</u>: $df = \omega^1_{\text{grad} f}$

<u>proof</u>: by defination $df = \sum_{\alpha=1}^{n} D_\alpha f\,dx^\alpha$

$\omega^1_{\text{grad} f}(P) = (\text{grad } f(P))^1 dx + (\text{grad } f(P))^2 dy + (\text{grad } f(P))^3 dz$

$= D_1 f(P)\,dx + D_2 f(P)\,dy + D_3 f(P)\,dz$

$= \sum_{\alpha=1}^{3} D_\alpha f(P)\,dx^\alpha = df(P)$

$\Rightarrow \omega^1_{\text{grad} f} = df$

(ii) curl $F = \nabla \times F = (D_2F^3 - D_3F^2)(e_1)_p$
$+ (D_3F^1 - D_1F^3)(e_2)_p$
$+ (D_1F^2 - D_2F^1)(e_3)_p$

$$\omega^2_{\text{curl} F} = (D_2F^3 - D_3F^2)\, dy \wedge dz + (D_3F^1 - D_1F^3)\, dz \wedge dx + (D_1F^2 - D_2F^1)\, dx \wedge dy$$

$$d(\omega^1_F) = d(F^1 dx + F^2 dy + F^3 dz)$$
$$= D_2F^1 dy \wedge dx + D_3F^1 dz \wedge dx + D_1F^2 dx \wedge dy + D_3F^2 dz \wedge dy + D_1F^3 dx \wedge dz + D_2F^3 dy \wedge dz$$
$$= (D_2F^3 - D_3F^2)\, dy \wedge dz + (D_3F^1 - D_1F^3)\, dz \wedge dx + (D_1F^2 - D_2F^1)\, dx \wedge dy$$
$$= \omega^2_{\text{curl} F}$$

(iii) $d(\omega^2_F) = d(F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy)$
$= D_1F^1 dx \wedge dy \wedge dz + D_2F^2 dx \wedge dy \wedge dz + D_3F^3 dx \wedge dy \wedge dz$

now $\text{Div} F = \sum_{i=1}^{n} D_i F^i$ $\quad = (\text{Div} F)\, dx \wedge dy \wedge dz$

so $\text{Div} F = D_1F^1 + D_2F^2 + D_3F^3$

6. (a) ω and η are closed differential forms, i.e
$d\omega = 0$
$d\eta = 0$

<u>To prove</u>: $\omega \wedge \eta$ is closed

<u>proof</u>: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
let ω be k form
$= 0 + (-1)^k (0)$
$= 0$
$\Rightarrow \omega \wedge \eta$ is closed

(b) if ω is closed and $\exists\, \xi$ s.t $d\xi = \eta$

<u>To prove</u>: $\omega \wedge \eta$ is exact

<u>proof</u>: we have to show that $\exists\, \tilde{\xi}$ s.t $d\tilde{\xi} = \omega \wedge \eta$
now
$$d(\xi \wedge \omega) = d(\xi) \wedge \omega + \underbrace{(-1)^{\circ} \xi \wedge (0)}_{0}$$
$$d(\xi \wedge \omega) = \eta \wedge \omega$$
then $\omega \wedge \eta = (-1)^{k_1 k_2} \eta \wedge \omega$
$= (-1)^{k_1 k_2} d(\xi \wedge \omega)$
$\omega \wedge \eta = d((-1)^{k_1 k_2} \xi \wedge \omega)$

so $\omega \wedge \eta$ is exact

$\omega \rightarrow k_1$ form

$\eta \rightarrow k_2$ form

Tutorial-12 :

1. $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ C^∞ function

F is C^∞ vector field on $\mathbb{R}^3$

$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$
s.t $\forall\, p \in F$
$F(P) \in \mathbb{R}^3_p$

$F(P) = F^1(P)(e_1)_p + F^2(P)(e_2)_p + F^3(P)(e_3)_p$
where $F^i(P) : \mathbb{R}^3 \longrightarrow \mathbb{R}$
$\in C^\infty$

(i) curl(grad f)

$\text{grad } f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$
s.t $P \in \mathbb{R}^3$
$(\text{grad } f)(P) \in \mathbb{R}^3_p$ $\quad f : \mathbb{R}^3 \to \mathbb{R}$

$(\text{grad} f)(P) = D_1 f(P)(e_1)_p + D_2 f(P)(e_2)_p + D_3 f(P)(e_3)_p$

F is a vector field given

$$(\text{curl } F)(P) = (D_2F^3 - D_3F^2)(P)(e_1)_p + (D_3F^1 - D_1F^3)(P)(e_2)_p + (D_1F^2 - D_2F^1)(P)(e_3)_p$$

$$\begin{aligned}(\text{curl } (\text{grad } f))(P) &= (D_2(D_3 f) - D_3(D_2 f))(P)(e_1)_p + \cdots + \cdots \\ &= (D_{3,2} f - D_{2,3} f)(P)(e_1)_p + \cdots + \cdots \\ &= 0 \quad \text{as } D_{3,2} f = D_{2,3} f \text{ as } f \text{ is } C^\infty \text{ given}\end{aligned}$$

$\Rightarrow \text{curl}(\text{grad } f) = 0$

(b) $\text{Div } F = \sum D_i F^i$
now, $F \Rightarrow \text{curl } F$

we have $(\text{curl } F)(P) = (D_2F^3 - D_3F^2)(P)(e_1)_p + (D_3F^1 - D_1F^3)(P)(e_2)_p + (D_1F^2 - D_2F^1)(P)(e_3)_p$

now, F^1 becomes $D_2F^3 - D_3F^2$
$\Rightarrow D_1F^1$ becomes $D_1(D_2F^3 - D_3F^2)$
$= D_{2,1}F^3 - D_{3,1}F^2$

similarly $D_2F^2 : D_{3,2}F^1 - D_{1,2}F^3$
$D_3F^3 : D_{1,3}F^2 - D_{2,3}F^1$

summing all we get

$$\begin{aligned}\text{div}(\text{curl } F) &= D_{2,1}F^3 - D_{1,2}F^3 + D_{3,2}F^1 - D_{2,3}F^1 + D_{1,3}F^2 - D_{3,1}F^2 \\ &= 0 \quad \text{as } D_{2,1} = D_{1,2} \text{ as } F^3 \text{ is } C^\infty\end{aligned}$$

2. F is C^∞ vector field on star shaped open set A $A \underset{\text{open}}{\subseteq} \mathbb{R}^3$

$\text{curl } F = 0$

<u>To prove</u>: $\exists f : A \to \mathbb{R}$ s.t $F = \text{grad } f$

proof: As A is star shaped, if $\exists\, \omega$ (k form on A) ($\omega \in \Omega^k(A)$) s.t $d\omega = 0$

then $\exists\, \eta \in \Omega^{k-1}(A)$ s.t $\omega = d\eta$

now as curl $F = 0$ 3

let $\omega_F^1 = \sum_{i=1} F^i dx^i \longrightarrow$ 1 form on A

then $d(\omega_F^1) = \omega^2_{\text{curl } F}$

as $\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy$

$\omega^2_{\text{curl } F} = 0$ as curl $F = 0$
$F^1 = 0, F^2 = 0, F^3 = 0$

$\Rightarrow d(\omega_F^1) = 0$

$\Rightarrow \exists\, \eta$ s.t $\omega_F^1 = d(\eta)$

where η is 0-form or $\eta : \mathbb{R}^3 \longrightarrow \mathbb{R}$

let $f = \eta$ then $\omega_F^1 = d(f)$

also, $\omega^1_{\text{grad} f} = df \Rightarrow \omega_F^1 = \omega^1_{\text{grad} f}$

from tutorial 11 - Q5 $\Rightarrow F = \text{grad} f$

$df = \omega^1_{\text{grad } f}$

$d(\omega^1_F) = \omega^2_{\text{curl } F}$

$d(\omega^2_F) = (\text{div} F) dx \wedge dy \wedge dz$

$\therefore \exists f : A \to \mathbb{R}$ s.t $\omega_F^1 = df = \omega^1_{\text{grad } F}$

as $d(\omega_F^1) = 0 = \omega^2_{\text{curl } F}$

<u>To prove</u>: If div $F = 0$, $\exists\, G$ a vectorfield on A s.t $F = \text{curl } G$

proof: as div $F = 0$

$d(\omega_F^2) = 0$

$\Rightarrow$ as A is starshaped $\exists\, \eta$ a 1 form s.t

$d\eta = \omega_F^2$

now as η is 1 form let $\eta = \omega^1_G$ 1 form

$= \sum_{i=1}^{3} G^i dx^i$

then $d(\omega^1_G) = \omega^2_F$

but $d(\omega^1_G) = \omega^2_{\text{curl } G}$
$= \omega^2_F$

$\Rightarrow \omega^2_{\text{curl } G} = \omega^2_F$

$\Rightarrow \text{curl } G = F$

so, $\exists\, G$ s.t $\text{curl } G = F$

4(b) $A = [0,1]^2 \quad c : A \longrightarrow \mathbb{R}^3$

$c(u,v) = (u, v, u^2+v^2+1)$

$\omega = x^2 x^3 dx^1 + x^1 dx^2 + dx^3$

$$\int_{\partial c} \omega = \int_c d\omega$$

$$d\omega = x^3 dx^2 \wedge dx^1 + x^2 dx^3 \wedge dx^1 + dx^1 \wedge dx^2 + 0$$
$$= (1-x^3) dx^1 \wedge dx^2 - x^2 dx^1 \wedge dx^3$$

$$\int_c d\omega = \int_c (1-x^3) dx^{i_1} \wedge dx^{i_2} - \int_c x^2 dx^{i_1} \wedge dx^{i_2}$$

$\det\left(\frac{d c^{i_1}\, c^{i_2}}{du\, dv}\right)$
$= \det\left(\frac{d c^1\, c^2}{du\, v}\right)$
$= \left|\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$

$= \det\left(\frac{c^1\, c^3}{du\, dv}\right)$
$= \left|\begin{bmatrix} 1 & 0 \\ 2u & 2v \end{bmatrix}\right|$

$$D_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{bmatrix}$$

$$\int_c d\omega = \int_{[0,1]^2} (1-x^3) \circ c \circ du\, dv$$
$$- \int_{[0,1]^2} (x^2) \circ c \circ 2v\, du\, dv$$

$$= \int_{[0,1]^2} (-u^2 - v^2)\, du\, dv - \int_{[0,1]^2} 2v^2\, du\, dv$$

$$= \int_{[0,1]^2} (-u^2 - 3v^2)\, du\, dv \quad = -\frac{1}{3} - \frac{3}{3} = -\frac{4}{3}$$

$\partial c = -c_{(1,0)} + c_{(1,1)} + c_{(2,0)} - c_{(2,1)}$

now $c_{(1,0)}(x) = c \circ I_{(1,0)}(x)$
$= (0, x, x^2+1)$

$c_{(1,1)}(x) = (1, x, x^2+2)$

$$C_{(2,0)}(x) = C \circ I_{(2,0)}(x)$$
$$= (x, 0, x^2+1)$$

$$C_{(2,1)}(x) = C \circ I_{(2,1)}(x)$$
$$= (x, 1, x^2+2)$$

$$\int_{\partial C} \omega = -\int_{C_{(1,0)}} \omega + \int_{C_{(1,1)}} \omega + \int_{C_{(2,0)}} \omega - \int_{C_{(2,1)}} \omega$$

$$= -\int_{C_{(1,0)}} x^2 x^3 dx^1 + x^1 dx^2 + dx^3 + \dots$$

$$= -\int_{[0,1]} x^1 \circ C_{(1,0)} \circ dx + (\cancel{2x}) dx$$

$$+ \int_{[0,1]} x^1 \circ C_{(1,1)} dx + (\cancel{2x}) dx$$

$$+ \int_{[0,1]} x^2 x^3 \circ C_{(2,0)} dx + \cancel{2x} dx$$

$$- \int_{[0,1]} x^2 x^3 \circ C_{(2,1)} dx + \cancel{2x} dx$$

$$= -\int_{[0,1]} 0 + \int_{[0,1]} dx + \int_{[0,1]} 0\, dx - \int_{[0,1]} (1)(x^2+2)\, dx$$

$$= 1 - 2 - \int_0^1 x^2 dx$$

$$= -1 - \frac{1}{3} = \frac{-4}{3}$$

4.(a) $\eta = x^2 dx^2 \wedge dx^3 + x^1 x^3 dx^2 \wedge dx^3$

$dx^1 \quad dx^3$

$d\eta = 0 \Rightarrow \eta = d\xi$ by poincare lemma

$$\xi = \frac{(x^2)^2}{2} dx^3 + \frac{(x^1)^2}{2} x^3 dx^3$$

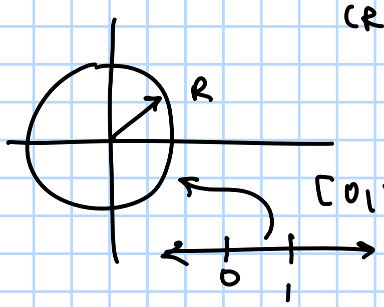
$$\int_C d\xi = \int_C \omega = \int_{\partial C} \xi$$

or just compute

3. $R>0$

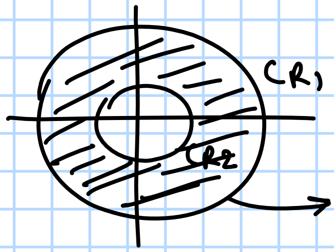
1-cube: $c_R : [0,1] \to \mathbb{R}^2 \setminus \{0\}$

$c_R(t) = (R\cos 2\pi t, R\sin 2\pi t)$



now $c : [0,1]^2 \to \mathbb{R}^2 \setminus \{0\}$ s.t $\partial c = c_{R_1} - c_{R_2}$

$[0,1]$ mapped to a circle



Intuition wise

$c : [0,1]^2 \to \mathbb{R}^2 \setminus \{0\}$

is the donut region

now $c_{R_1} - c_{R_2} = \delta c$ Boundary of donut

let $c(s,t) = [R_1(1-s) + R_2(s)] \times \underbrace{e^{2\pi i t}}_{(\cos 2\pi t, \sin 2\pi t)}$

so let $c(s,t) = ([R_1(1-s) + R_2 s]\cos 2\pi t, [R_1(1-s) + R_2(s)]\sin 2\pi t)$

then $\partial c = -c_{(1,0)} + c_{(1,1)} + c_{(2,0)} - c_{(2,1)}$

$= -(c \circ I^2_{(1,0)}) + c \circ I^2_{(1,1)}$

$= -c(0,x) + c(1,x) + c(x,0) - c(x,1)$

$= -R_1 e^{2\pi i x} + R_2 e^{2\pi i x} + 0 + 0$

$= c_{R_2} - c_{R_1}$

just switch R_1 and R_2

4. (a) $\eta = x^2 dx^2 \wedge dx^3 + x^1 x^3 dx^1 \wedge dx^3$

$c(u,v) = (u, v, u^2+v^2+1)$

$\int_c \eta = \int_c x^2 dx^2 \wedge dx^3 + \int_c x^1 x^3 dx^1 \wedge dx^3$

$= \int_c x^2 \circ c \begin{bmatrix} 0 & 1 \\ 2u & 2v \end{bmatrix} du\,dv + \int_c x^1 x^3 \circ c \begin{bmatrix} 1 & 0 \\ 2u & 2v \end{bmatrix} du\,dv$

$= \int_{[0,1]^2} v(-2u)\,du\,dv + \int_{[0,1]^2} (u)(u^2+v^2+1)(2v)\,du\,dv$

$$= \int_0^1\int_0^1 (-2\cancel{uv} + 2vu^3 + 2uv^3 + 2\cancel{uv})\,du\,dv$$

$$= 2\int_0^1\int_0^1 vu^3 + uv^3\,du\,dv$$

$$= 2\times\cancel{2}\times\frac{1}{4}\times\frac{1}{\cancel{2}}$$

$$= \frac{1}{2}$$

(b) $\omega = x^2x^3dx^1 + x^1dx^2 + dx^3$

$\int_{\partial c}\omega = \int_c d\omega$ verification

$A = [0,1]^2 \quad c : A \longrightarrow \mathbb{R}^3$

$c(u,v) = (u, v, u^2+v^2+1)$

$\omega = x^2x^3dx^1 + x^1dx^2 + dx^3$

$$\int_{\partial c}\omega = \int_c d\omega$$

$$d\omega = x^3dx^2\wedge dx^1 + x^2dx^3\wedge dx^1 + dx^1\wedge dx^2 + 0$$
$$= (1-x^3)dx^1\wedge dx^2 - x^2dx^1\wedge dx^3$$

$$\int_c d\omega = \int_c (1-x^3)dx^{i_1}\wedge dx^{i_2} - \int_c x^2dx^{i_1}\wedge dx^{i_2}$$

$\det\left(\frac{\partial c^{i_1}c^{i_2}}{\partial u\,\partial v}\right)$

$= \det\left(\frac{\partial c^1c^2}{\partial u\,v}\right)$

$= \left\|\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\|$

$= \det\left(\frac{c^1c^3}{\partial u\,\partial v}\right)$

$= \left\|\begin{bmatrix}1 & 0\\ 2u & 2v\end{bmatrix}\right\|$

$$D_c = \begin{bmatrix}1 & 0\\ 0 & 1\\ 2u & 2v\end{bmatrix}$$

$$\int_c d\omega = \int_{[0,1]^2}(1-x^3)\circ c\circ du\,dv - \int_{[0,1]^2}(x^2)\circ c\circ 2v\,du\,dv$$

$$= \int_{[0,1]^2}(-u^2-v^2)\,du\,dv - \int_{[0,1]^2}2v^2\,du\,dv$$

$$= \int_{[0,1]^2}(-u^2-3v^2)\,du\,dv$$

$$= -\frac{1}{3} - \frac{3}{3} = -\frac{4}{3}$$

$$\partial c = -c_{(1,0)} + c_{(1,1)} + c_{(2,0)} - c_{(2,1)}$$

now
$$c_{(1,0)}(x) = c \circ I_{(1,0)}(x) = (0, x, x^2+1)$$
$$c_{(1,1)}(x) = (1, x, x^2+2)$$

$$c_{(2,0)}(x) = c \circ I_{(2,0)}(x) = (x, 0, x^2+1)$$

$$c_{(2,1)}(x) = c \circ I_{(2,1)}(x) = (x, 1, x^2+2)$$

$$\int_{\partial c} \omega = -\int_{c_{(1,0)}} \omega + \int_{c_{(1,1)}} \omega + \int_{c_{(2,0)}} \omega - \int_{c_{(2,1)}} \omega$$

$$= -\int_{c_{(1,0)}} x^2x^3dx^1 + x^1dx^2 + dx^3 + \dots$$

$$= -\int_{[0,1]} x^1 \circ c_{(1,0)} \circ dx + (\cancel{2x})dx$$
$$+ \int_{[0,1]} x^1 \circ c_{(1,1)} dx + (\cancel{2x})dx$$
$$+ \int_{[0,1]} x^2x^3 \circ c_{(2,0)} dx + \cancel{2x}dx$$
$$- \int_{[0,1]} x^2x^3 \circ c_{(2,1)} dx + \cancel{2x} dx$$

$$= -\int_{[0,1]} 0 + \int_{[0,1]} dx + \int_{[0,1]} 0\, dx - \int_{[0,1]} (1)(x^2+2)\, dx$$

$$= 1 - 2 - \int_0^1 x^2 dx$$

$$= -1 - \frac{1}{3} = \frac{-4}{3}$$

$$\text{so } \int_{\partial c} \omega = \int_c d\omega$$

5. $A = [0,1]^3$

$c : A \longrightarrow \mathbb{R}^4$

$c(s,t,u) = (s, u, t, (2u-t)^2)$

3-cube in $\mathbb{R}^4$

$\omega = 3$-form

$$\omega = x^1 dx^1 \wedge dx^4 \wedge dx^3 + 2x^2x^3 dx^1 \wedge dx^2 \wedge dx^3$$
$$= -x^1 dx^1 \wedge dx^3 \wedge dx^4 + 2x^2x^3 dx^1 \wedge dx^2 \wedge dx^3$$

now $\int_c \omega = \int_c -x^1 dx^1 \wedge dx^3 \wedge dx^4 + \int_c 2x^2 x^3 dx^1 \wedge dx^2 \wedge dx^3$

$$= -\int_{[0,1]^3} x^1 \circ c \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2(2u-t)(-1) & 2(2u-t)(2) \end{pmatrix} du\,dt\,ds$$

$$+ \int_{[0,1]^3} 2x^2 x^3 \circ c \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} du\,dt\,ds$$

$$= \int_{[0,1]^3} -(s)(2)(2u-t)(2)\,du\,dt\,ds + \int_{[0,1]^3} 2ut\,du\,dt\,ds$$

$$= \int_{[0,1]^3} 4[ts - 2us] + 2ut\,du\,dt\,ds$$

$$= \int_{[0,1]^3} 4ts - 8us + 2ut\,du\,dt\,ds$$

$$= \cancel{4}\frac{1}{\cancel{2}}\frac{1}{\cancel{2}} - \overset{2}{\cancel{8}}\frac{1}{2}\frac{1}{\cancel{2}} + \cancel{2}\frac{1}{\cancel{2}}\frac{1}{\cancel{2}}\frac{1}{2}$$

$$= 1 - 2 + \frac{1}{2}$$

$$= -\frac{1}{2}$$